1.8 Relating Trigonometric Functions

Learning Objectives

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.
- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.
- State the domain and range of each trig function.
- State the sign of a trig function, given the quadrant in which an angle lies.
- State the Pythagorean identities and use these identities to find values of trig functions.

Reciprocal identities

The first set of identities we will establish are the reciprocal identities. A **reciprocal** of a fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{or} \quad \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \quad \text{or} \quad \tan\theta = \frac{1}{\cot\theta}$$

Example 1: Find the value of each expression using a reciprocal identity.

a. $\cos\theta = .3, \sec\theta = ?$

b. $\cot\theta = \frac{4}{3}, \tan\theta = ?$

Solution:

a. $\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = \frac{10}{3}$.

b. $\tan\theta = \frac{3}{4}$

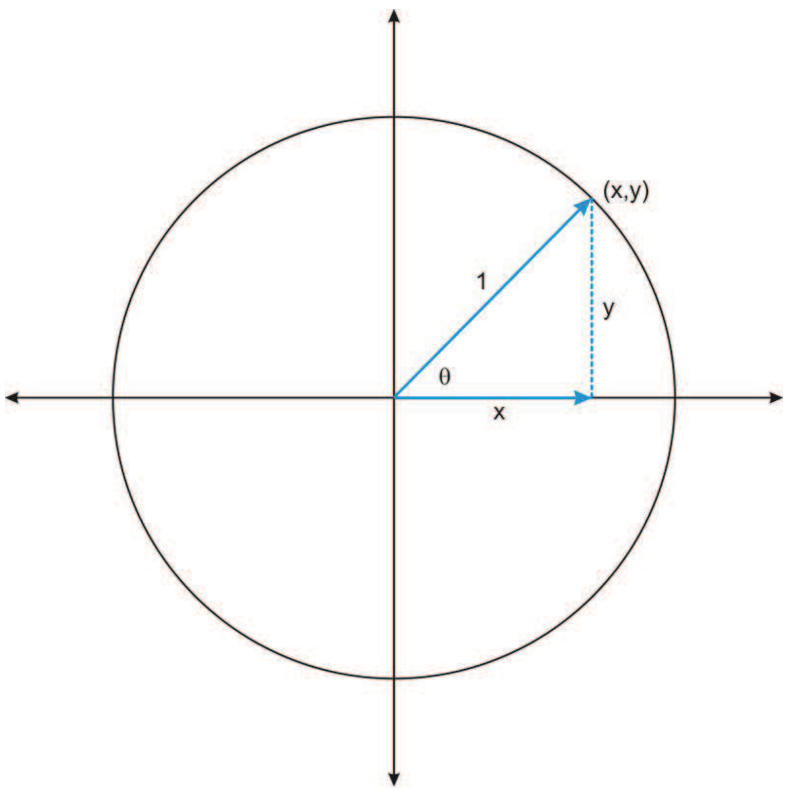
These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

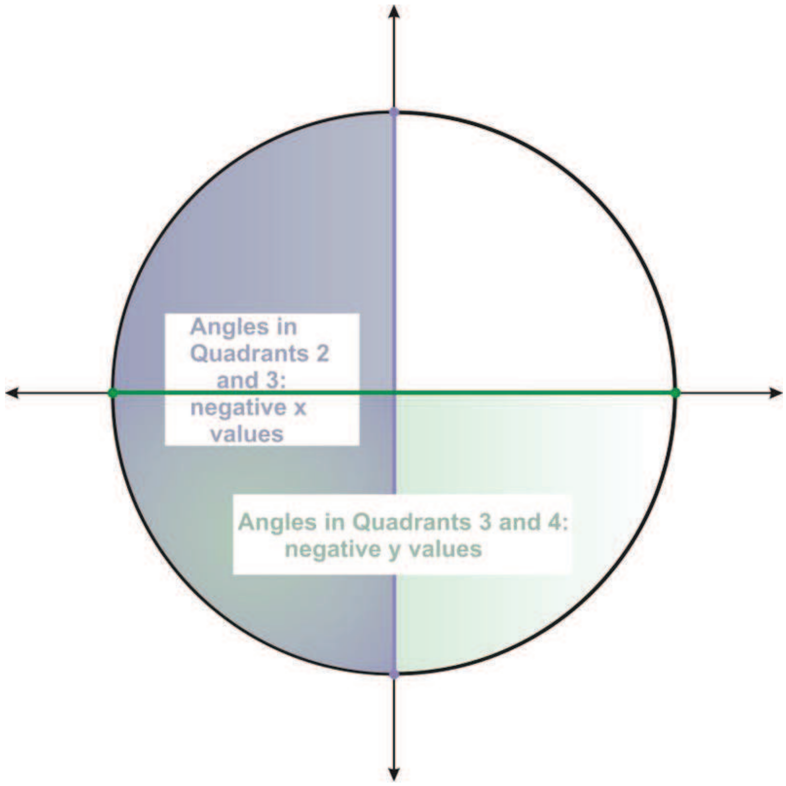
Domain, Range, and Signs of Trig Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, R. We can determine the range of the functions if we think about the fact that the sine of an angle is the $y-$coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $x-$coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.



In this right triangle, x and y are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative $y-$coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative $x-$coordinate.



In either case, the minimum value is -1. For example, $\cos 180^\circ = -1$ and $\sin 270^\circ = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

TABLE 1.5:

	Domain	Range
Sine	$\theta = R$	$-1 \leq y \leq 1$
Cosine	$\theta = R$	$-1 \leq y \leq 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

TABLE 1.6:

	Domain	Range
Cosecant	$\theta \varepsilon R, \theta \neq 0, 180, 360 \ldots$	$\csc\theta \leq -1$ or $\csc\theta \geq 1$
Secant	$\theta \varepsilon R, \theta \neq 90, 270, 450 \ldots$	$\sec\theta \leq -1$ or $\sec\theta \geq 1$

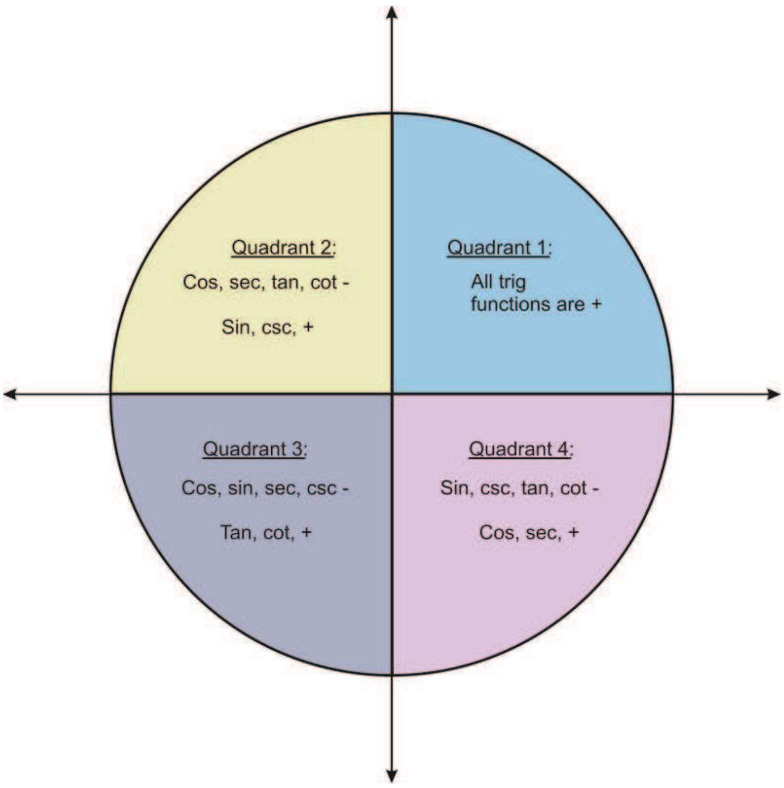
Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan\theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an $x-$coordinate of $0 : 90^\circ, 270^\circ$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a $y-$coordinate of $0: 0°, 180°, 360°$, etc.

TABLE 1.7:

Function	Domain	Range
Tangent	$\theta\varepsilon R, \theta \neq 90, 270, 450\ldots$	All reals
Cotangent	$\theta\varepsilon R, \theta \neq 0, 180, 360\ldots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.



An easy way to remember this is "All Students Take Calculus." Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which trig functions are positive and where.

Example 2: State the sign of each expression.

a. $\cos 100°$

b. $\csc 220°$

c. $\tan 370°$

Solution:

a. The angle $100°$ is in the second quadrant. Therefore the $x-$coordinate is negative and so $\cos 100°$ is negative.

b. The angle 220° is in the third quadrant. Therefore the $y-$coordinate is negative. So the sine, and the cosecant are negative.

c. The angle 370° is in the first quadrant. Therefore the tangent value is positive.

So far we have considered relationships between pairs of functions: the six trig functions can be grouped in pairs as reciprocals. Now we will consider relationships among three trig functions.

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities above. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example 3: If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

Solution: $\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example 4: Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solution:

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

This is also an identity that you can use to find the value of the cotangent function, given values of sine and cosine. Both of the quotient identities will also be useful in chapter 3, in which you will prove other identities.

Cofunction Identities and Reflection

These identities relate to the problems you did in section 1.3. Recall, #3 and #4 from the review questions, where $\sin X = \cos Z$ and $\cos X = \sin Z$, where X and Z were complementary angles. These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$
$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Example 5: Find the value of each trig function.

a. $\cos 120^\circ$

b. $\cos(-120^\circ)$

c. $\sin 135^\circ$

d. $\sin(-135^\circ)$

Solution: Because these angles have reference angles of 60° and 45°, the values are:

a. $\cos 120^\circ = -\frac{1}{2}$

b. $\cos(-120^\circ) = \cos 240^\circ = -\frac{1}{2}$

c. $\sin 135^\circ = \frac{\sqrt{2}}{2}$

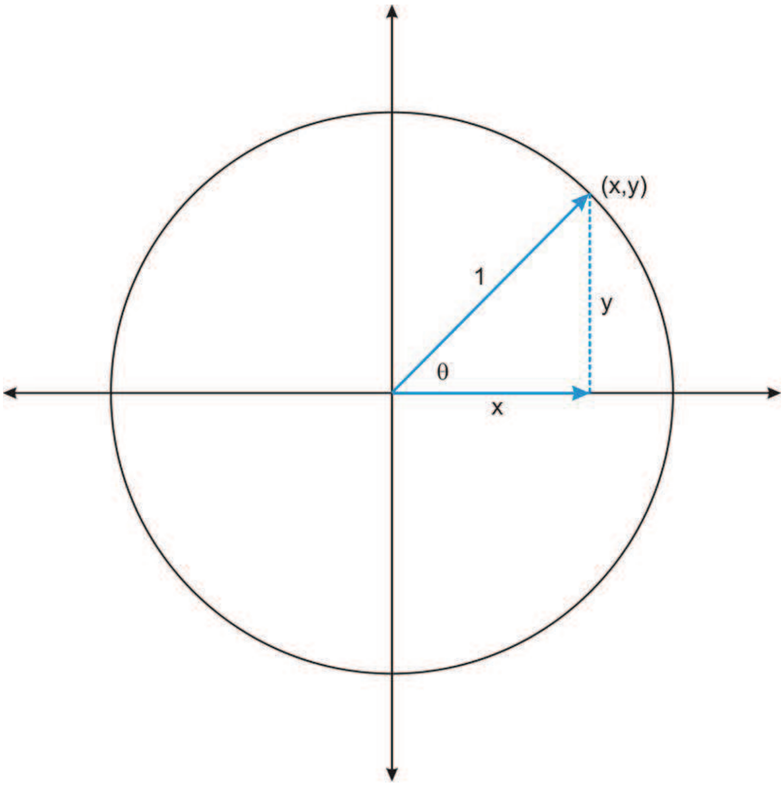
d. $\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2}$

These values show us that sine and cosine also reflect over the x axis. This allows us to generate three more identities.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Pythagorean Identities

The final set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In previous lessons we used the Pythagorean Theorem to find the sides of right triangles. Consider once again the way that we defined the trig functions in 1.3. Let's look at the unit circle:



The legs of the right triangle are x, and y. The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ *means* $(\cos\theta)^2$ *and* $\sin^2\theta$ *means* $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example 6: If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\begin{aligned}
\cos^2\theta+\sin^2\theta &= 1\\
\left(\frac{1}{4}\right)^2+\sin^2\theta &= 1\\
\frac{1}{16}+\sin^2\theta &= 1\\
\sin^2\theta &= 1-\frac{1}{16}\\
\sin^2\theta &= \frac{15}{16}\\
\sin\theta &= \pm\sqrt{\frac{15}{16}}\\
\sin\theta &= \pm\frac{\sqrt{15}}{4}
\end{aligned}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example 7: Use the identity $\cos^2\theta+\sin^2\theta = 1$ to show that $\cot^2\theta+1 = \csc^2\theta$

Solution:

$\cos^2\theta+\sin^2\theta = 1$	Divide both sides by $\sin^2\theta$.
$\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$	
$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$	$\frac{\sin^2\theta}{\sin^2\theta} = 1$
$\frac{\cos^2\theta}{\sin^2\theta}+1 = \frac{1}{\sin^2\theta}$	
$\frac{\cos\theta}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}+1 = \frac{1}{\sin\theta}\times\frac{1}{\sin\theta}$	Write the squared functions in terms of their factors.
$\cot\theta\times\cot\theta+1 = \csc\theta\times\csc\theta$	Use the quotient and reciprocal identities.
$\cot^2\theta+1 = \csc^2\theta$	Write the functions as squared functions.

Points to Consider

1. How do you know if an equation is an identity? *HINT: you could consider using a the calculator and graphing a related function, or you could try to prove it mathematically.*
2. How can you verify the domain or range of a function?

Review Questions

1. Use reciprocal identities to give the value of each expression.
   a. $\sec\theta = 4, \cos\theta = ?$
   b. $\sin\theta = \frac{1}{3}, \csc\theta = ?$
2. In the lesson, the range of the cosecant function was given as: $\csc\theta \leq -1$ or $\csc\theta \geq 1$.
   a. Use a calculator to fill in the table below. Round values to 4 decimal places.
   b. Use the values in the table to explain in your own words what happens to the values of the cosecant function as the measure of the angle approaches 0 degrees.
   c. Explain what this tells you about the range of the cosecant function.
   d. Discuss how you might further explore values of the sine and cosecant to better understand the range of the cosecant function.

TABLE 1.8:

Angle	Sin	Csc
10		
5		
1		
0.5		
0.1		
0		
-.1		
-.5		
-1		
-5		
-10		

3. In the lesson the domain of the secant function were given: Domain: $\theta\varepsilon^\circ, \theta \neq 90, 270, 450\ldots$ Explain why certain values are excluded from the domain.
4. State the quadrant in which each angle lies, and state the sign of each expression
   a. $\sin 80^\circ$
   b. $\cos 200^\circ$
   c. $\cot 325^\circ$
   d. $\tan 110^\circ$
5. If $\cos\theta = \frac{6}{10}$ and $\sin\theta = \frac{8}{10}$, what is the value of $\tan\theta$?
6. Use quotient identities to explain why the tangent and cotangent function have positive values for angles in the third quadrant.
7. If $\sin\theta = 0.4$, what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.
8. If $\cot\theta = 2$, what is the value of $\csc\theta$? Assume that θ is an angle in the first quadrant.
9. Show that $1 + \tan^2\theta = \sec^2\theta$.
10. Explain why it is necessary to state the quadrant in which the angle lies for problems such as #7.

Review Answers

1. 1. $\frac{1}{4}$

    2. $\frac{3}{1} = 3$

2. (a)

TABLE 1.9:

Angle	Sin	Csc
10	.1737	5.759
5	.0872	11.4737
1	.0175	57.2987
0.5	.0087	114.5930
0.1	.0018	572.9581
0	0	undefined
-.1	-.0018	-572.9581
-.5	-.0087	-114.5930
-1	-.0175	-57.2987
-5	-.0872	-11.4737
-10	-.1737	-5.759

(b) As the angle gets smaller and smaller, the cosecant values get larger and larger.

(c) The range of the cosecant function does not have a maximum, like the sine function. The values get larger and larger.

(d) Answers will vary. For example, if we looked at values near 90 degrees, we would see the cosecant values get smaller and smaller, approaching 1.

3. The values 90, 270, 450, etc, are excluded because they make the function undefined.
    1. Quadrant 1; positive
    2. Quadrant 3; negative
    3. Quadrant 4; negative
    4. Quadrant 2; negative
4. $\frac{8}{6} = \frac{4}{3}$
5. The ratio of sine and cosine will be positive in the third quadrant because sine and cosine are both negative in the third quadrant.
6. $\cos\theta \approx .92$
7. $\csc\theta = \sqrt{5}$
8.

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\ 1 + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\ 1 + \tan^2\theta &= \sec^2\theta\end{aligned}$$

9. Using the Pythagorean identities results in a quadratic equation and will have two solutions. Stating that the angle lies in a particular quadrant tells you which solution is the actual value of the expression. In #7, the angle is in the first quadrant, so both sine and cosine must be positive.