Right Triangles and an Introduction to Trigonometry

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CHAPTER 1 Right Triangles and an Introduction to Trigonometry

CHAPTER OUTLINE

1.1 The Pythagorean Theorem

Introduction

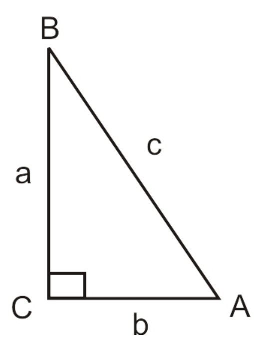
Right triangles play an integral part in the study of trigonometry. It is from right triangles that the basic definitions of the trigonometric functions are formed. In this chapter we will explore right triangles and their properties. Through this, we will introduce the six basic trig functions and the unit circle.

Learning Objectives

- Recognize and use the Pythagorean Theorem.
- Recognize basic Pythagorean Triples.
- Use the Distance Formula.

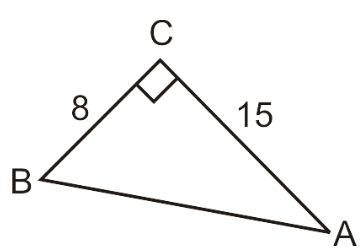
The Pythagorean Theorem

From Geometry, recall that the Pythagorean Theorem is $a^2+b^2=c^2$ where a and b are the legs of a right triangle and c is the hypotenuse. Also, the side opposite the angle is lower case and the angle is upper case. For example, angle A is opposite side a.



The Pythagorean Theorem is used to solve for the sides of a right triangle.

Example 1: Use the Pythagorean Theorem to find the missing side.

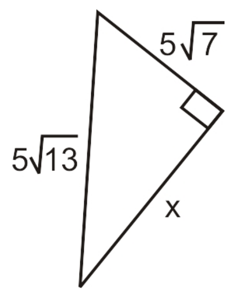


Solution: $a = 8,\ b = 15$, we need to find the hypotenuse.

$$\begin{aligned} 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 \\ 17 &= c \end{aligned}$$

Notice, we do not include -17 as a solution because a negative number cannot be a side of a triangle.

Example 2: Use the Pythagorean Theorem to find the missing side.



Solution: Use the Pythagorean Theorem to find the missing leg.

$$\begin{aligned} \left(5\sqrt{7}\right)^2 + x^2 &= \left(5\sqrt{13}\right)^2 \\ 25 \cdot 7 + x^2 &= 25 \cdot 13 \\ 175 + x^2 &= 325 \\ x^2 &= 150 \\ x &= 5\sqrt{6} \end{aligned}$$

Pythagorean Triples

Pythagorean Triples are sets of whole numbers for which the Pythagorean Theorem holds true. The most well-known triple is 3, 4, 5. This means, that 3 and 4 are the lengths of the legs and 5 is the hypotenuse. *The largest length is always the hypotenuse*. If we were to multiply any triple by a constant, this new triple would still represent sides of a right triangle. Therefore, 6, 8, 10 and 15, 20, 25, among countless others, would represent sides of a right triangle.

Example 3: Determine if the following lengths are Pythagorean Triples.

a. 7, 24, 25

b. 9, 40, 41

c. 11, 56, 57

Solution: Plug each set of numbers into the Pythagorean Theorem.

a.

$$\begin{aligned} 7^2 + 24^2 &\stackrel{?}{=} 25^2 \\ 49 + 576 &= 625 \\ 625 &= 625 \end{aligned}$$

Yes, 7, 24, 25 is a Pythagorean Triple and sides of a right triangle.

b.

$$\begin{aligned} 9^2 + 40^2 &\stackrel{?}{=} 41^2 \\ 81 + 1600 &= 1681 \\ 1681 &= 1681 \end{aligned}$$

Yes, 9, 40, 41 is a Pythagorean Triple and sides of a right triangle.

c.

$$\begin{aligned} 11^2 + 56^2 &\stackrel{?}{=} 57^2 \\ 121 + 3136 &= 3249 \\ 3257 &\neq 3249 \end{aligned}$$

No, 11, 56, 57 do not represent the sides of a right triangle.

Converse of the Pythagorean Theorem

Using the technique from Example 3, we can determine if sets of numbers are acute, right or obtuse triangles. Examples 3a and 3b were both right triangles because the two sides equaled each other and made the Pythagorean Theorem true. However in Example 3c, the two sides were not equal. Because $3257 > 3249$, we can say that 11, 56, and 57 are the sides of an acute triangle. To help you visualize this, think of an equilateral triangle with sides of length 5. We know that this is an acute triangle. If you plug in 5 for each number in the Pythagorean Theorem we get $5^2 + 5^2 = 5^2$ and $50 > 25$. Therefore, if $a^2 + b^2 > c^2$, then lengths a, b, and c make up an acute triangle. Conversely, if $a^2 + b^2 < c^2$, then lengths a, b, and c make up the sides of an obtuse triangle. *It is important to note that the length "c'' is always the longest.*

Example 4: Determine if the following lengths make an acute, right or obtuse triangle.

a. 5, 6, 7

b. 5, 10, 14

c. 12, 35, 37

Solution: Plug in each set of lengths into the Pythagorean Theorem.

a.

$$\begin{aligned} 5^2 + 6^2 &\ ?\ 7^2 \\ 25 + 36 &\ ?\ 49 \\ 61 &> 49 \end{aligned}$$

Because $61 > 49$, this is an acute triangle.

b.

$$\begin{aligned}5^2+10^2 &\ ?\ 14^2\\ 25+100 &\ ?\ 196\\ 125 &< 196\end{aligned}$$

Because $125 < 196$, this is an obtuse triangle.

c.

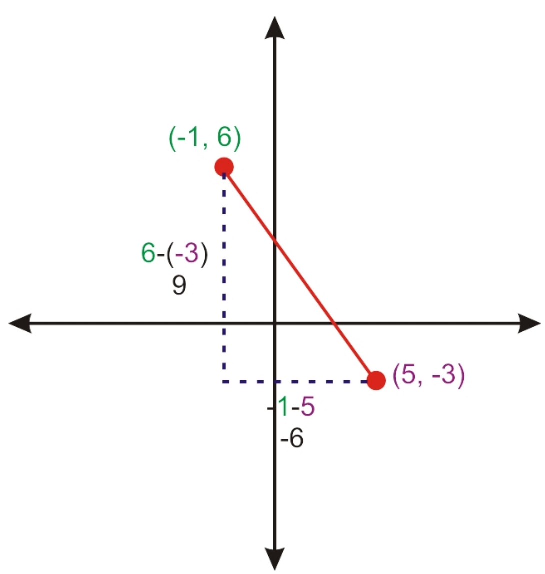
$$\begin{aligned}12^2+35^2 &\ ?\ 37^2\\ 144+1225 &\ ?\ 1369\\ 1369 &= 1369\end{aligned}$$

Because the two sides are equal, this is a right triangle.

NOTE: All of the lengths in Example 4 represent the lengths of the sides of a triangle. Recall the Triangle Inequality Theorem from geometry which states: The length of a side in a triangle is less than the sum of the other two sides. For example, 4, 7 and 13 cannot be the sides of a triangle because $4+7$ is not greater than 13.

The Distance Formula

An application of the Pythagorean Theorem is to find the distance between two points. Consider the points (-1, 6) and (5, -3). If we plot them on a grid, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

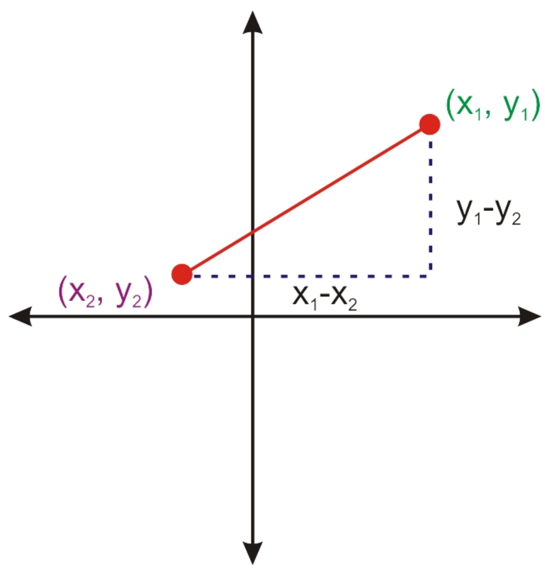


Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$\begin{aligned}9^2+(-6)^2&=d^2\\81+36&=d^2\\117&=d^2\\\sqrt{117}&=d\\3\sqrt{13}&=d\end{aligned}$$

Notice, that the $x-$values were subtracted from each other to find the horizontal distance and the $y-$values were subtracted from each other to find the vertical distance. If this process is generalized for two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, the Distance Formula is derived.

$$(x_1-x_2)^2+(y_1-y_2)^2=d^2$$



This is the Pythagorean Theorem with the vertical and horizontal differences between $(x_1,\ y_1)$ and $(x_2,\ y_2)$. Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=d$$

This is the Distance Formula. The following example shows how to apply the distance formula.

Example 5: Find the difference between the two points.

a. (4, 2) and (-9, 5)

b. (-10, 3) and (0, -15)

Solution: Plug each pair of points into the distance formula.

a.

$$\begin{aligned} d &= \sqrt{(4-(-9))^2+(2-5)^2} \\ &= \sqrt{13^2+(-3)^2} \\ &= \sqrt{169+9} \\ &= \sqrt{178} \end{aligned}$$

b.

$$\begin{aligned} d &= \sqrt{(-10-0)^2+(3-(-15))^2} \\ &= \sqrt{(-10)^2+(18)^2} \\ &= \sqrt{100+324} \\ &= \sqrt{424} = 2\sqrt{106} \end{aligned}$$

Points to Consider

- Does the Pythagorean Theorem apply to all real numbers?
- Can a Pythagorean Triple have irrational numbers in the set?
- What is the difference between the Distance Formula and the Pythagorean Theorem?

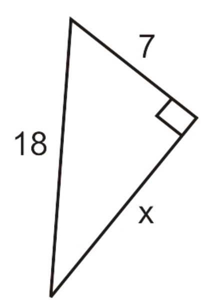
Review Questions

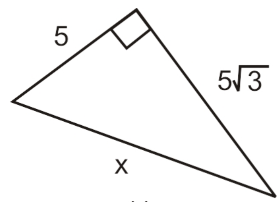
Determine if the lengths below represent the sides of a right triangle. If not, state if the triangle is acute or obtuse.

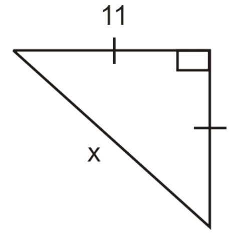
1. 6, 9, 13
2. 9, 10, 11
3. 16, 30, 34
4. 20, 23, 40
5. 11, 16, 29
6. $2\sqrt{6}$, $6\sqrt{3}$, $2\sqrt{33}$

Find the missing side of each right triangle below. Leave the answer in simplest radical form.

7.



8. 

9. 

10. The general formula for a Pythagorean Triple is $n^2 - m^2, 2nm, n^2 + m^2$ where n and m are natural numbers. Use the Pythagorean Theorem to prove this is true.
11. Find the distance between the pair of points.

  a. (5, -6) and (18, 3)
  b. $\left(\sqrt{3}, -\sqrt{2}\right)$ and $\left(-2\sqrt{3}, 5\sqrt{2}\right)$

Review Answers

1. $6^2 + 9^2 ? 13^2 \rightarrow 36 + 81 ? 169 \rightarrow 117 < 169$ The triangle is obtuse.
2. $9^2 + 10^2 ? 11^2 \rightarrow 81 + 100 ? 121 \rightarrow 181 > 121$ The triangle is acute.
3. $16^2 + 30^2 ? 34^2 \rightarrow 256 + 900 ? 1156 \rightarrow 1156 = 1156$ This is a right triangle.
4. $20^2 + 23^2 ? 40^2 \rightarrow 400 + 529 ? 1600 \rightarrow 929 < 1600$ The triangle is obtuse.
5. These lengths cannot make up the sides of a triangle. $11 + 16 < 29$
6. $\left(2\sqrt{6}\right)^2 + \left(6\sqrt{3}\right)^2 ? \left(2\sqrt{33}\right)^2 \rightarrow (4 \cdot 6) + (36 \cdot 3) ? (4 \cdot 33) \rightarrow 24 + 108 ? 132 \rightarrow 132 = 132$ This is a right triangle.
7.

$$\begin{aligned}
7^2 + x^2 &= 18^2 \\
49 + x^2 &= 324 \\
x^2 &= 275 \\
x &= \sqrt{275} = 5\sqrt{11}
\end{aligned}$$

8.

$$\begin{aligned}
5^2 + \left(5\sqrt{3}\right)^2 &= x^2 \\
25 + (25 \cdot 3) &= x^2 \\
25 + 75 &= x^2 \\
100 &= x^2 \\
10 &= x
\end{aligned}$$

9. Both legs are 11.

$$\begin{aligned}11^2+11^2&=x^2\\121+121&=x^2\\242&=x^2\\\sqrt{242}&=x\\11\sqrt{2}&=x\end{aligned}$$

10. Plug $n^2-m^2, 2nm, n^2+m^2$ into the Pythagorean Theorem.

$$\begin{aligned}(n^2-m^2)^2+(2nm)^2&=(n^2+m^2)^2\\n^4-2n^2m^2+m^4+4n^2m^2&=n^4+2n^2m^2+m^4\\-2n^2m^2+4n^2m^2&=2n^2m^2\\4n^2m^2&=4n^2m^2\end{aligned}$$

11. (a) (5, -6) and (18, 3)

$$\begin{aligned}d=&\sqrt{(5-18)^2+(-6-3)^2}\\=&\sqrt{(-13)^2+(-9)^2}\\=&\sqrt{169+81}\\=&\sqrt{250}\\=&5\sqrt{10}\end{aligned}$$

(b) $\left(\sqrt{3},\ -\sqrt{2}\right)$ and $\left(-2\sqrt{3},\ 5\sqrt{2}\right)$

$$\begin{aligned}d=&\sqrt{\left(\sqrt{3}-\left(-2\sqrt{3}\right)\right)^2+\left(-\sqrt{2}-5\sqrt{2}\right)^2}\\=&\sqrt{\left(3\sqrt{3}\right)^2+\left(-6\sqrt{2}\right)^2}\\=&\sqrt{(9\cdot3)+(36\cdot2)}\\=&\sqrt{27+72}\\=&\sqrt{99}=3\sqrt{11}\end{aligned}$$

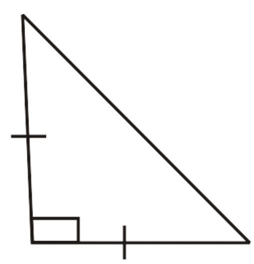
1.2 Special Right Triangles

Learning Objectives

- Recognize special right triangles.
- Use the special right triangle ratios to solve special right triangles.

Special Right Triangle #1: Isosceles Right Triangle

An isosceles right triangle is an isosceles triangle and a right triangle. This means that it has two congruent sides and one right angle. Therefore, the two congruent sides must be the legs.



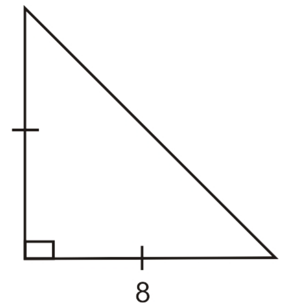
Because the two legs are congruent, we will call them both a and the hypotenuse c. Plugging both letters into the Pythagorean Theorem, we get:

$$\begin{aligned}
a^2 + a^2 &= c^2 \\
2a^2 &= c^2 \\
\sqrt{2a^2} &= \sqrt{c^2} \\
a\sqrt{2} &= c
\end{aligned}$$

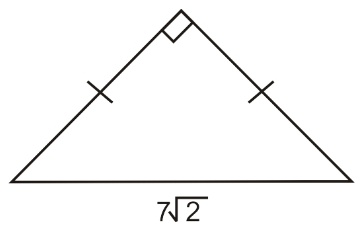
From this we can conclude that the hypotenuse length is the length of a leg multiplied by $\sqrt{2}$. Therefore, we only need one of the three lengths to determine the other two lengths of the sides of an isosceles right triangle. The ratio is usually written $x : x : x\sqrt{2}$, where x is the length of the legs and $x\sqrt{2}$ is the length of the hypotenuse.

Example 1: Find the lengths of the other two sides of the isosceles right triangles below.

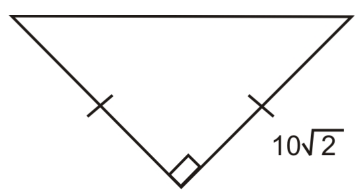
a.



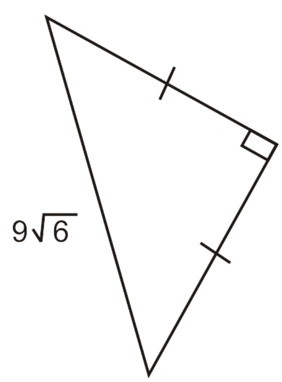
b.



c.



d.



Solution:

a. If a leg has length 8, by the ratio, the other leg is 8 and the hypotenuse is $8\sqrt{2}$.

b. If the hypotenuse has length $7\sqrt{2}$, then both legs are 7.

c. Because the leg is $10\sqrt{2}$, then so is the other leg. The hypotenuse will be $10\sqrt{2}$ multiplied by an additional $\sqrt{2}$.

$$10\sqrt{2}\cdot\sqrt{2} = 10\cdot 2 = 20$$

d. In this problem set $x\sqrt{2} = 9\sqrt{6}$ because $x\sqrt{2}$ is the hypotenuse portion of the ratio.

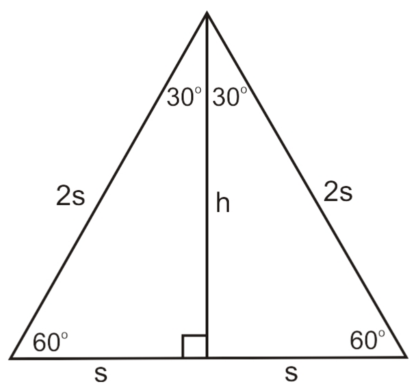
$$x\sqrt{2} = 9\sqrt{6}$$
$$x = \frac{9\sqrt{6}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{9\sqrt{12}}{2} = \frac{18\sqrt{3}}{2} = 9\sqrt{3}$$

So, the length of each leg is $9\sqrt{3}$.

What are the angle measures in an isosceles right triangle? Recall that the sum of the angles in a triangle is 180° and there is one 90° angle. Therefore, the other two angles add up to 90°. Because this is an isosceles triangle, these two angles are equal and 45° each. Sometimes an isosceles right triangle is also referred to as a $45-45-90$ triangle.

Special Right Triangle #2: 30-60-90 Triangle

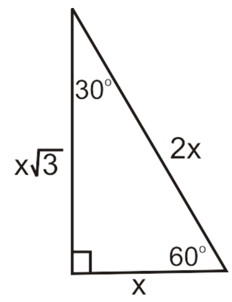
$30-60-90$ refers to each of the angles in this special right triangle. To understand the ratios of the sides, start with an equilateral triangle with an altitude drawn from one vertex.



Recall from geometry, that an altitude, h, cuts the opposite side directly in half. So, we know that one side, the hypotenuse, is $2s$ and the shortest leg is s. Also, recall that the altitude is a perpendicular and angle bisector, which is why the angle at the top is split in half. To find the length of the longer leg, use the Pythagorean Theorem:

$$\begin{aligned} s^2 + h^2 &= (2s)^2 \\ s^2 + h^2 &= 4s^2 \\ h^2 &= 3s^2 \\ h &= s\sqrt{3} \end{aligned}$$

From this we can conclude that the length of the longer leg is the length of the short leg multiplied by $\sqrt{3}$ or $s\sqrt{3}$. Just like the isosceles right triangle, we now only need one side in order to determine the other two in a $30-60-90$ triangle. The ratio of the three sides is written $x : x\sqrt{3} : 2x$, where x is the shortest leg, $x\sqrt{3}$ is the longer leg and $2x$ is the hypotenuse.

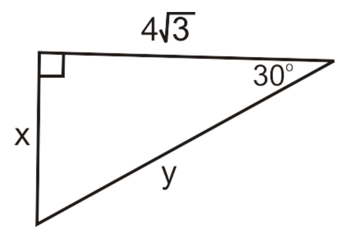


Notice, that the shortest side is *always* opposite the smallest angle and the longest side is *always* opposite 90°.

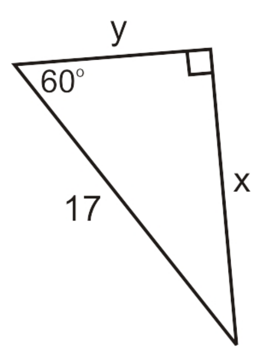
If you look back at the Review Questions from the last section we now recognize #8 as a $30-60-90$ triangle.

Example 2: Find the lengths of the two missing sides in the $30-60-90$ triangles.

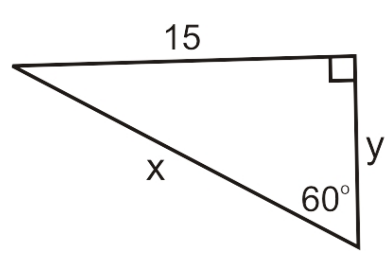
a.



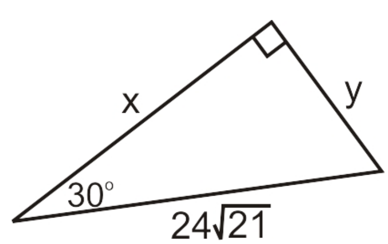
b.



c.



d.



Solution: Determine which side in the $30-60-90$ ratio is given and solve for the other two.

a. $4\sqrt{3}$ is the longer leg because it is opposite the 60°. So, in the $x : x\sqrt{3} : 2x$ ratio, $4\sqrt{3} = x\sqrt{3}$, therefore $x = 4$ and $2x = 8$. The short leg is 4 and the hypotenuse is 8.

b. 17 is the hypotenuse because it is opposite the right angle. In the $x : x\sqrt{3} : 2x$ ratio, $17 = 2x$ and so the short leg is $\frac{17}{2}$ and the long leg is $\frac{17\sqrt{3}}{2}$.

c. 15 is the long leg because it is opposite the 60°. Even though 15 does not have a radical after it, we can still set it equal to $x\sqrt{3}$.

$$x\sqrt{3} = 15$$
$$x = \frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3} \quad \text{So, the short leg is } 5\sqrt{3}.$$

Multiplying $5\sqrt{3}$ by 2, we get the hypotenuse length, which is $10\sqrt{3}$.

d. $24\sqrt{21}$ is the length of the hypotenuse because it is opposite the right angle. Set it equal to $2x$ and solve for x to get the length of the short leg.

$$2x = 24\sqrt{21}$$
$$x = 12\sqrt{21}$$

To find the length of the longer leg, we need to multiply $12\sqrt{21}$ by $\sqrt{3}$.

$$12\sqrt{21} \cdot \sqrt{3} = 12\sqrt{3 \cdot 3 \cdot 7} = 36\sqrt{7}$$

The length of the longer leg is $36\sqrt{7}$.

Be careful when doing these problems. You can always check your answers by finding the decimal approximations of each side. For example, in 2d, short leg $= 12\sqrt{21} \approx 54.99$, long leg $= 36\sqrt{7} \approx 95.25$ and the hypotenuse $= 24\sqrt{21} \approx 109.98$. This is an easy way to double-check your work and verify that the hypotenuse is the longest side.

Using Special Right Triangle Ratios

Special right triangles are the basis of trigonometry. The angles 30°, 45°, 60° and their multiples have special properties and significance in the unit circle (sections 1.5 and 1.6). Students are usually required to memorize these two ratios because of their importance.

First, let's compare the two ratios, so that we can better distinguish the difference between the two. For a $45-45-90$ triangle the ratio is $x : x : x\sqrt{2}$ and for a $30-60-90$ triangle the ratio is $x : x\sqrt{3} : 2x$. An easy way to tell the difference between these two ratios is the isosceles right triangle has two congruent sides, so its ratio has the $\sqrt{2}$, whereas the $30-60-90$ angles are all divisible by 3, so that ratio includes the $\sqrt{3}$. Also, if you are ever in doubt or forget the ratios, you can always use the Pythagorean Theorem. The ratios are considered a short cut.

Example 3: Determine if the sets of lengths below represent special right triangles. If so, which one?

a. $8\sqrt{3} : 24 : 16\sqrt{3}$

b. $\sqrt{5} : \sqrt{5} : \sqrt{10}$

c. $6\sqrt{7} : 6\sqrt{21} : 12$

Solution:

a. Yes, this is a $30-60-90$ triangle. If the short leg is $x = 8\sqrt{3}$, then the long leg is $8\sqrt{3} \cdot \sqrt{3} = 8 \cdot 3 = 24$ and the hypotenuse is $2 \cdot 8\sqrt{3} = 16\sqrt{3}$.

b. Yes, this is a $45-45-90$ triangle. The two legs are equal and $\sqrt{5} \cdot \sqrt{2} = \sqrt{10}$, which would be the length of the hypotenuse.

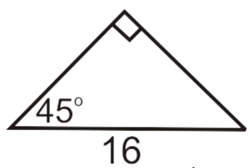
c. No, this is not a special right triangle, nor a right triangle. The hypotenuse should be $12\sqrt{7}$ in order to be a $30-60-90$ triangle.

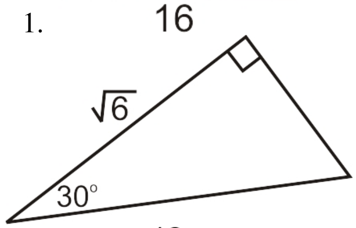
Points to Consider

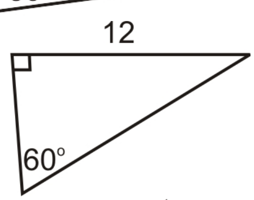
- What is the difference between Pythagorean triples and special right triangle ratios?
- Why are these two ratios considered "special"?

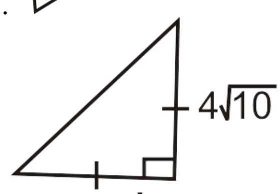
Review Questions

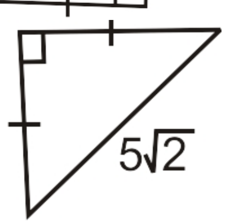
Solve each triangle using the special right triangle ratios.

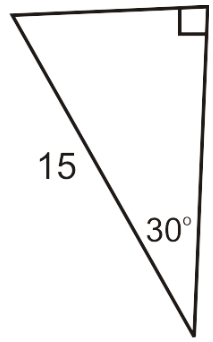
1. 

2. 

3. 

4. 

5. 

6. 

7. A square window has a diagonal of 6 ft. To the nearest hundredth, what is the height of the window?
8. Pablo has a rectangular yard with dimensions 10 ft by 20 ft. He is decorating the yard for a party and wants to hang lights along both diagonals of his yard. How many feet of lights does he need? Round your answer to the nearest foot.
9. Can $2:2:2\sqrt{3}$ be the sides of a right triangle? If so, is it a special right triangle?
10. Can $\sqrt{5}:\sqrt{15}:2\sqrt{5}$ be the sides of a right triangle? If so, is it a special right triangle?

Review Answers

1. Each leg is $\frac{16}{\sqrt{2}} = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$.
2. Short leg is $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}$ and hypotenuse is $2\sqrt{2}$.
3. Short leg is $\frac{12}{\sqrt{3}} = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$ and hypotenuse is $8\sqrt{3}$.
4. The hypotenuse is $4\sqrt{10}\cdot\sqrt{2} = 4\sqrt{20} = 8\sqrt{5}$.
5. Each leg is $\frac{5\sqrt{2}}{\sqrt{2}} = 5$.
6. The short leg is $\frac{15}{2}$ and the long leg is $\frac{15\sqrt{3}}{2}$.
7. If the diagonal of a square is 6 ft, then each side of the square is $\frac{6}{\sqrt{2}}$ or $3\sqrt{2} \approx 4.24\ ft$.
8. These are not dimensions for a special right triangle, so to find the diagonal (both are the same length) do the Pythagorean Theorem:

$$\begin{aligned} 10^2 + 20^2 &= d^2 \\ 100 + 400 &= d^2 \\ \sqrt{500} &= d \\ 10\sqrt{5} &= d \end{aligned}$$

So, if each diagonal is $10\sqrt{5}$, two diagonals would be $20\sqrt{5} \approx 45\ ft$. Pablo needs 45 ft of lights for his yard.
9. $2:2:2\sqrt{3}$ does not fit into either ratio, so it is not a special right triangle. To see if it is a right triangle, plug these values into the Pythagorean Theorem:

$$\begin{aligned} 2^2 + 2^2 &= \left(2\sqrt{3}\right)^2 \\ 4 + 4 &= 12 \\ 8 &< 12 \end{aligned}$$

this is not a right triangle, it is an obtuse triangle.
10. $\sqrt{5}:\sqrt{15}:2\sqrt{5}$ is a $30-60-90$ triangle. The long leg is $\sqrt{5}\cdot\sqrt{3} = \sqrt{15}$ and the hypotenuse is $2\sqrt{5}$.

1.3 Basic Trigonometric Functions

Learning Objectives

- Find the values of the six trigonometric functions for angles in right triangles.

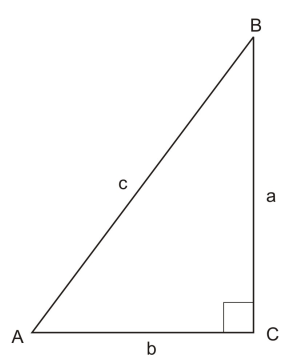
Introduction

Consider a situation in which you are building a ramp for wheelchair access to a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

Solving this kind of problem requires trigonometry. The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define six trigonometric functions. For each of these functions, the elements of the domain are angles. We will define these functions in two ways: first, using right triangles, and second, using angles of rotation. Once we have defined these functions, we will be able to solve problems like the one above.

The Sine, Cosine, and Tangent Functions

The first three trigonometric functions we will work with are the sine, cosine, and tangent functions. As noted above, the elements of the domains of these functions are angles. We can define these functions in terms of a right triangle: The elements of the range of the functions are particular ratios of sides of triangles.



We define the sine function as follows: For an acute angle x in a right triangle, the *sinx* is equal to the ratio of the side opposite of the angle over the hypotenuse of the triangle. For example, using this triangle, we have: $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Since all right triangles with the same acute angles are similar, this function will produce the same ratio, no matter which triangle is used. Thus, it is a well-defined function.

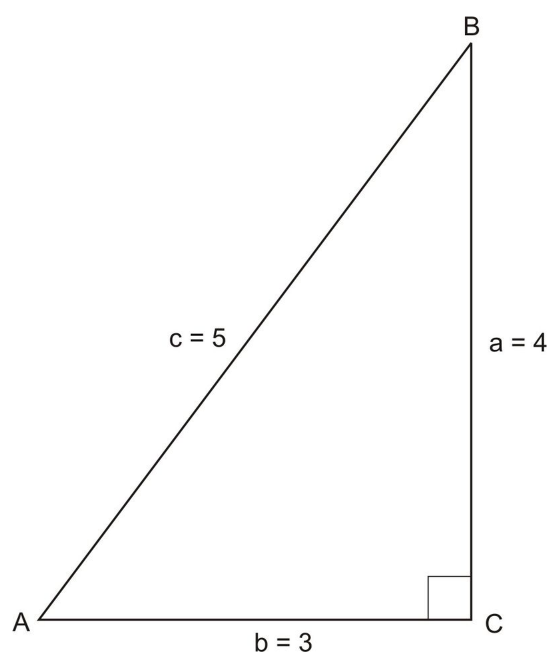
Similarly, the cosine of an angle is defined as the ratio of the side adjacent (next to) the angle over the hypotenuse of the triangle. Using this triangle, we have: $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Finally, the tangent of an angle is defined as the ratio of the side opposite the angle to the side adjacent to the angle. In the triangle above, we have: $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these functions. First, keep in mind that the abbreviations $sinx, cosx$, and $tanx$ are just like $f(x)$. They simply stand for specific kinds of functions. Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $sinx$ it is still pronounced *sine*, with a long "i.″ When we write $cosx$, we still say co-sine. And when we write $tanx$, we still say tangent.

We can use these definitions to find the sine, cosine, and tangent values for angles in a right triangle.

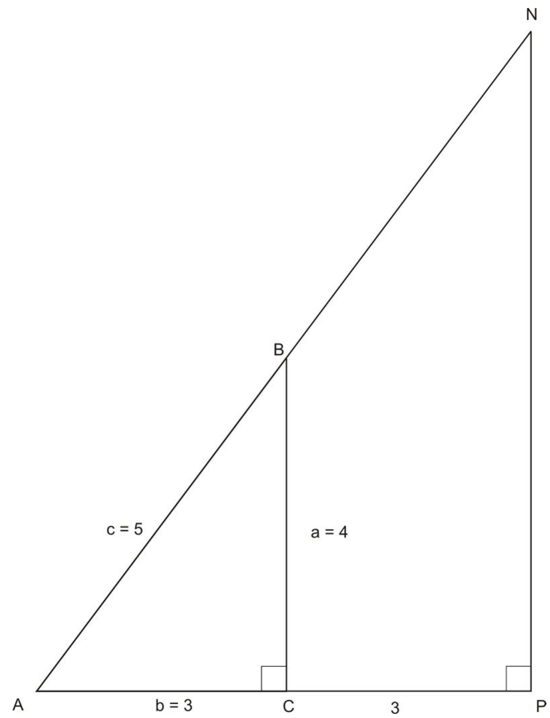
Example 1: Find the sine, cosine, and tangent of $\angle A$:



Solution:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{3}$$

One of the reasons that these functions will help us solve problems is that these ratios will always be the same, as long as the angles are the same. Consider for example, a triangle similar to triangle ABC.



If CP has length 3, then side AP of triangle NAP is 6. Because NAP is similar to ABC, side NP has length 8. This means the hypotenuse AN has length 10. (This can be shown either by using Pythagorean Triples or the Pythagorean Theorem.)

If we use triangle NAP to find the sine, cosine, and tangent of angle A, we get:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{8}{6} = \frac{4}{3}$$

Also notice that the tangent function is the same as the slope of the hypotenuse. $\tan A = \frac{4}{3}$, which is the same as $\frac{rise}{run}$ or $\frac{\text{change in } y}{\text{change in } x}$. The $\tan B$ does not equal the slope because it is the reciprocal of $tanA$.

Example 2: Find $\sin B$ using triangle ABC and triangle NAP.

Solution:

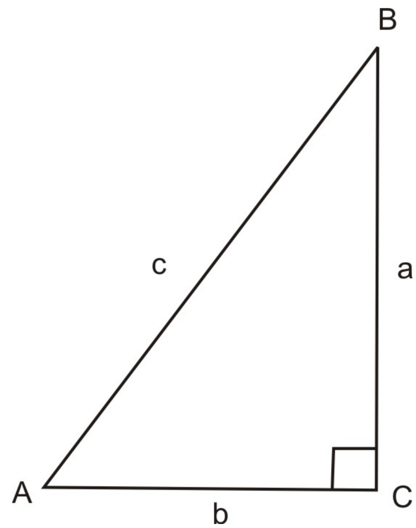
Using triangle $ABC : \sin B = \frac{3}{5}$

Using triangle $NAP : \sin B = \frac{6}{10} = \frac{3}{5}$

An easy way to remember the ratios of the sine, cosine, and tangent functions is SOH-CAH-TOA. Sine $= \frac{\text{Opposite}}{\text{Hypotenuse}}$, Cosine $= \frac{\text{Adjacent}}{\text{Hypotenuse}}$, Tangent $= \frac{\text{Opposite}}{\text{Adjacent}}$.

Secant, Cosecant, and Cotangent Functions

We can define three more functions also based on a right triangle. They are the reciprocals of sine, cosine and tangent.

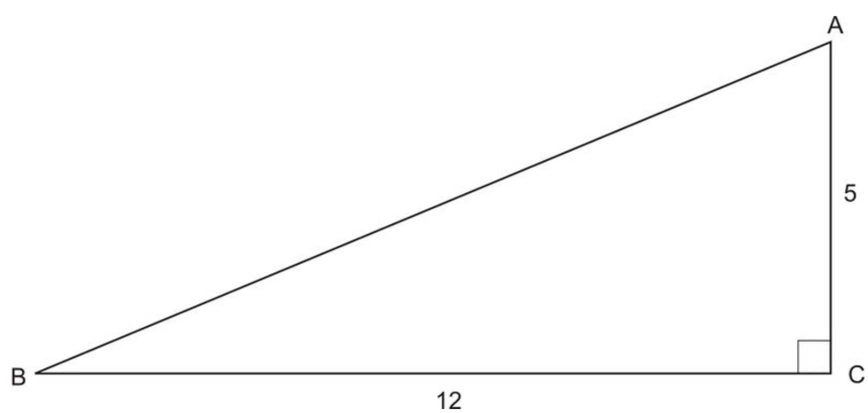


If $\sin A = \frac{a}{c}$, then the definition of cosecant, or csc, is $\csc A = \frac{c}{a}$.

If $\cos A = \frac{b}{c}$, then the definition of secant, or sec, is $\sec A = \frac{c}{b}$.

If $\tan A = \frac{a}{b}$, then the definition of cotangent, or cot, is $\cot A = \frac{b}{a}$.

Example 3: Find the secant, cosecant, and cotangent of angle B.



Solution:

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$\begin{aligned} 5^2 + 12^2 &= H^2 \\ 25 + 144 &= H^2 \\ 169 &= H^2 \\ H &= 13 \end{aligned}$$

Now we can find the secant, cosecant, and cotangent of angle B:

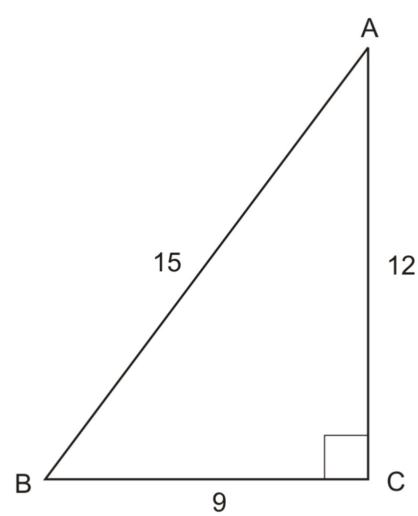
$$\sec B = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$
$$\csc B = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$
$$\cot B = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

Points to Consider

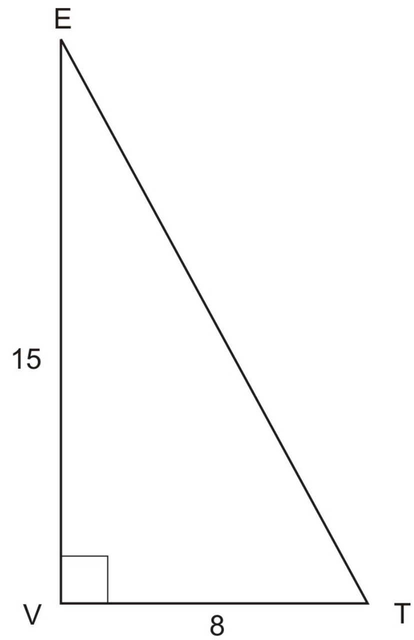
- Do you notice any similarities between the sine of one angle and the cosine of the other, in the same triangle?

Review Questions

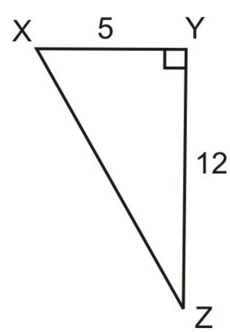
1. Find the values of the six trig functions of angle A.



2. Consider triangle VET below. Find the length of the hypotenuse and values of the six trig functions of angle T.



3. Consider the right triangle below.



   a. Find the hypotenuse.
   b. Find the six trigonometric functions of $\angle X$.
   c. Find the six trigonometric functions of $\angle Z$.

4. Looking back at #3, are any functions of $\angle X$ equal to any of the functions of $\angle Z$? If so, which ones? Do you think this could be generalized for ANY pair of acute angles in the same right triangle (also called complements)?
5. Consider an isosceles right triangle with legs of length 2. Find the sine, cosine and tangent of both acute angles.
6. Consider an isosceles right triangle with legs of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #5.
7. Consider a $30-60-90$ triangle with hypotenuse of length 10. Find the sine, cosine and tangent of both acute angles.
8. Consider a $30-60-90$ triangle with short leg of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #7.
9. Consider a right triangle, ABC. If $\sin A = \frac{9}{41}$, find the length of the third side.

Review Answers

1. $\sin A = \frac{9}{15} = \frac{3}{5}, \cos A = \frac{12}{15} = \frac{4}{5}, \tan A = \frac{9}{12} = \frac{3}{4}, \csc A = \frac{15}{9} = \frac{5}{3}, \sec A = \frac{15}{12} = \frac{5}{4}, \cot A = \frac{12}{9} = \frac{4}{3}$

2. The hypotenuse is 17 $\left(\sqrt{15^2+8^2} = \sqrt{225+64} = \sqrt{289} = 17\right)$.

$$\sin T = \frac{15}{17}, \cos T = \frac{8}{17}, \tan T = \frac{15}{8}, \csc T = \frac{17}{15}, \sec T = \frac{17}{8}, \cot T = \frac{8}{15}$$

   1. The hypotenuse is 13 $\left(\sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13\right)$.
   2. $\sin X = \frac{12}{13}, \cos X = \frac{5}{13}, \tan X = \frac{12}{5}, \csc X = \frac{13}{12}, \sec X = \frac{13}{5}, \cot X = \frac{5}{12}$
   3. $\sin Z = \frac{5}{13}, \cos Z = \frac{12}{13}, \tan Z = \frac{5}{12}, \csc Z = \frac{13}{5}, \sec Z = \frac{13}{12}, \cot Z = \frac{12}{5}$

3. From #3, we can conclude that $\sin X = \cos Z, \cos X = \sin Z, \tan X = \cot Z, \cot X = \tan Z, \csc X = \sec Z$ and $\sec X = \csc Z$. Yes, this can be generalized for all complements.
4. The hypotenuse is $2\sqrt{2}$. Each angle is 45°, so the sine, cosine, and tangent are the same for both angles.

$$\sin 45^\circ = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \cos 45^\circ = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \tan 45^\circ = \frac{2}{2} = 1$$

5. If the legs are length x, then the hypotenuse is $x\sqrt{2}$. For 45°, the sine, cosine, and tangent are:

$$\sin 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \cos 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \tan 45^\circ = \frac{x}{x} = 1$$

   This tells us that regardless of the length of the sides of an isosceles right triangle, the sine, cosine and tangent of 45° are always the same.
6. If the hypotenuse is 10, then the short leg is 5 and the long leg is $5\sqrt{3}$. Recall, that 30° is opposite the short side, or 5, and 60° is opposite the long side, or $5\sqrt{3}$.

$$\sin 30^\circ = \frac{5}{10} = \frac{1}{2}, \cos 30^\circ = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}, \tan 30^\circ = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\sin 60^\circ = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{5}{10} = \frac{1}{2}, \tan 60^\circ = \frac{5\sqrt{3}}{5} = \sqrt{3}$$

7. If the short leg is x, then the long leg is $x\sqrt{3}$ and the hypotenuse is $2x$. 30° is opposite the short side, or x, and 60° is opposite the long side, or $x\sqrt{3}$.

$$\sin 30^\circ = \frac{x}{2x} = \frac{1}{2}, \cos 30^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}, \tan 30^\circ = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\sin 60^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{x}{2x} = \frac{1}{2}, \tan 60^\circ = \frac{x\sqrt{3}}{x} = \sqrt{3}$$

   This tells us that regardless of the length of the sides of a $30-60-90$ triangle, the sine, cosine and tangent of 30° and 60° are always the same. Also, $\sin 30^\circ = \cos 60^\circ$ and $\cos 30^\circ = \sin 60^\circ$.
8. If $\sin A = \frac{9}{41}$, then the opposite side is $9x$ (some multiple of 9) and the hypotenuse is $41x$. Therefore, working with the Pythagorean Theorem would give us the length of the other leg. Also, we could notice that this is a Pythagorean Triple and the other leg is $40x$.

1.4 Solving Right Triangles

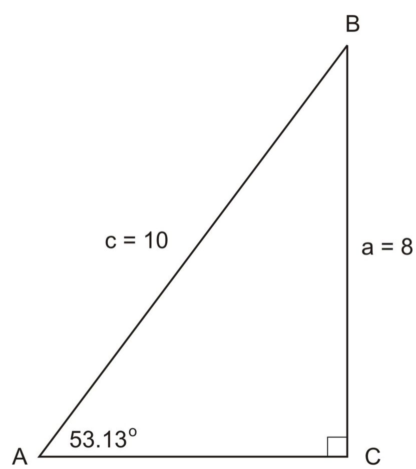
Learning Objectives

- Solve right triangles.
- Find the area of any triangle using trigonometry.
- Solve real-world problems that require you to solve a right triangle.
- Find angle measures using inverse trigonometric functions.

Solving Right Triangles

You can use your knowledge of the Pythagorean Theorem and the six trigonometric functions to solve a right triangle. Because a right triangle is a triangle with a 90 degree angle, solving a right triangle requires that you find the measures of one or both of the other angles. How you solve will depend on how much information is given. The following examples show two situations: a triangle missing one side, and a triangle missing two sides.

Example 1: Solve the triangle shown below.



Solution:

We need to find the lengths of all sides and the measures of all angles. In this triangle, two of the three sides are given. We can find the length of the third side using the Pythagorean Theorem:

$$
\begin{aligned}
8^2 + b^2 &= 10^2 \\
64 + b^2 &= 100 \\
b^2 &= 36 \\
b &= \pm 6 \Rightarrow b = 6
\end{aligned}
$$

(You may have also recognized the "Pythagorean Triple," 6, 8, 10, instead of carrying out the Pythagorean Theorem.)

You can also find the third side using a trigonometric ratio. Notice that the missing side, b, is adjacent to $\angle A$, and the hypotenuse is given. Therefore we can use the cosine function to find the length of b:

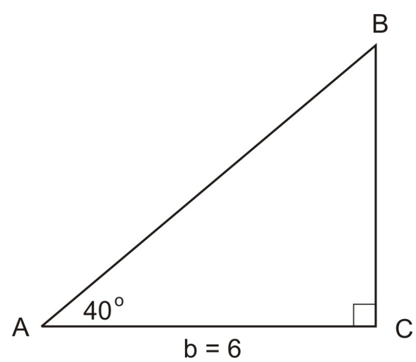
$$\begin{aligned}\cos 53.13^\circ &= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{10}\\ 0.6 &= \frac{b}{10}\\ b &= 0.6(10) = 6\end{aligned}$$

We could also use the tangent function, as the opposite side was given. It may seem confusing that you can find the missing side in more than one way. The point is, however, not to create confusion, but to show that you must look at what information is missing, and choose a strategy. Overall, when you need to identify one side of the triangle, you can either use the Pythagorean Theorem, or you can use a trig ratio.

To solve the above triangle, we also have to identify the measures of all three angles. Two angles are given: 90 degrees and 53.13 degrees. We can find the third angle using the Triangle Sum Theorem, $180-90-53.13 = 36.87^\circ$.

Now let's consider a triangle that has two missing sides.

Example 2: Solve the triangle shown below.



Solution:

In this triangle, we need to find the lengths of two sides. We can find the length of one side using a trig ratio. Then we can find the length of the third side by using a trig ratio with the same given information, not the side we solved for. This is because the side we found is an *approximation* and would not yield the most accurate answer for the other missing side. *Only use the given information when solving right triangles.*

We are given the measure of angle A, and the length of the side adjacent to angle A. If we want to find the length of the hypotenuse, c, we can use the cosine ratio:

$$\begin{aligned}\cos 40^\circ &= \frac{adjacent}{hypotenuse} = \frac{6}{c}\\ \cos 40^\circ &= \frac{6}{c}\\ c\cos 40^\circ &= 6\\ c &= \frac{6}{\cos 40^\circ} \approx 7.83\end{aligned}$$

If we want to find the length of the other leg of the triangle, we can use the tangent ratio. This will give us the most accurate answer because we are not using approximations.

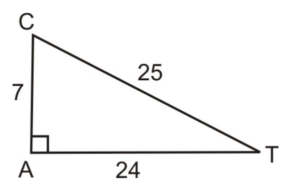
$$\tan 40^\circ = \frac{opposite}{adjacent} = \frac{a}{6}$$
$$a = 6\tan 40^\circ \approx 5.03$$

Now we know the lengths of all three sides of this triangle. In the review questions, you will verify the values of c and a using the Pythagorean Theorem. Here, to finish solving the triangle, we only need to find the measure of $\angle B: 180 - 90 - 40 = 50^\circ$

Notice that in both examples, one of the two non-right angles was given. If neither of the two non-right angles is given, you will need a new strategy to find the angles.

Inverse Trigonometric Functions

Consider the right triangle below.



From this triangle, we know how to determine all six trigonometric functions for both $\angle C$ and $\angle T$. From any of these functions we can also find the value of the angle, using our graphing calculators. If you look back at #7 from 1.3, we saw that $\sin 30^\circ = \frac{1}{2}$. If you type 30 into your graphing calculator and then hit the $\boxed{\text{SIN}}$ button, the calculator yields 0.5. (Make sure your calculator's mode is set to degrees.)

Conversely, with the triangle above, we know the trig ratios, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. These functions are located above the SIN, COS, and TAN buttons on the calculator. To access this function, press 2^{nd} and the appropriate button and the measure of the angle appears on the screen.

$\cos T = \frac{24}{25} \rightarrow \cos^{-1}\frac{24}{25} = T$ from the calculator we get

```
cos-1(24/25)
         16.26020471
```

Example 3: Find the angle measure for the trig functions below.

a. $\sin x = 0.687$

b. $\tan x = \frac{4}{3}$

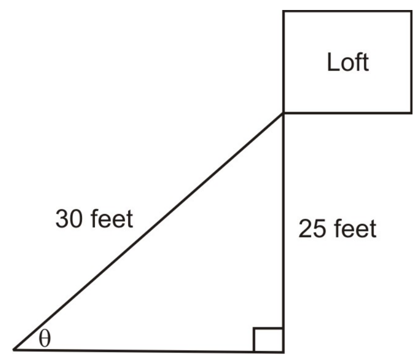
Solution: Plug into calculator.

a. $\sin^{-1} 0.687 = 43.4^\circ$

b. $\tan^{-1}\frac{4}{3} = 53.13^\circ$

Example 4: You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy and to move it manually down a ladder would take too much time and effort. You decide to devise a make shift conveyor belt made of bed sheets that you will attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?

Solution:



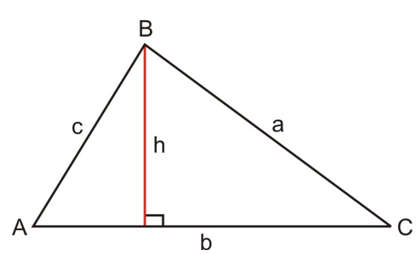
From the picture, we need to use the inverse sine function.

$$\begin{aligned}\sin\theta &= \frac{25\ feet}{30\ feet}\\ \sin\theta &= 0.8333\\ \sin^{-1}(\sin\theta) &= \sin^{-1}0.8333\\ \theta &= 56.4^\circ\end{aligned}$$

The sheets should be anchored at an angle of 56.4°.

Finding the Area of a Triangle

In Geometry, you learned that the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height, or altitude. Now that you know the trig ratios, this formula can be changed around, using sine.



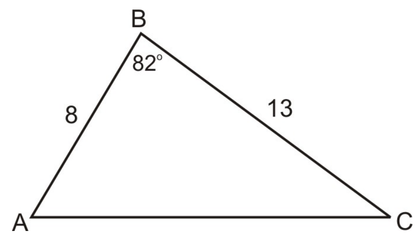
Looking at the triangle above, you can use sine to determine $h, \sin C = \frac{h}{a}$. So, solving this equation for h, we have $a\sin C = h$. Substituting this for h, we now have a new formula for area.

$$A = \frac{1}{2}ab\sin C$$

What this means is you do not need the height to find the area anymore. All you now need is two sides and the angle between the two sides, called the included angle.

Example 5: Find the area of the triangle.

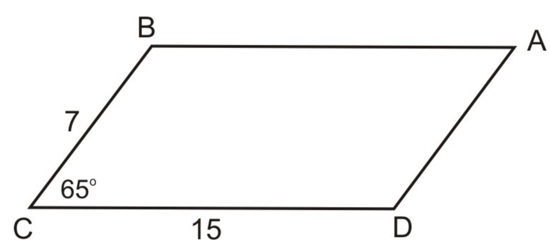
a.



Solution: Using the formula, $A = \frac{1}{2}\, ab \sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 8 \cdot 13 \cdot \sin 82^\circ \\ &= 4 \cdot 13 \cdot 0.990 \\ &= 51.494 \end{aligned}$$

Example 6: Find the area of the parallelogram.

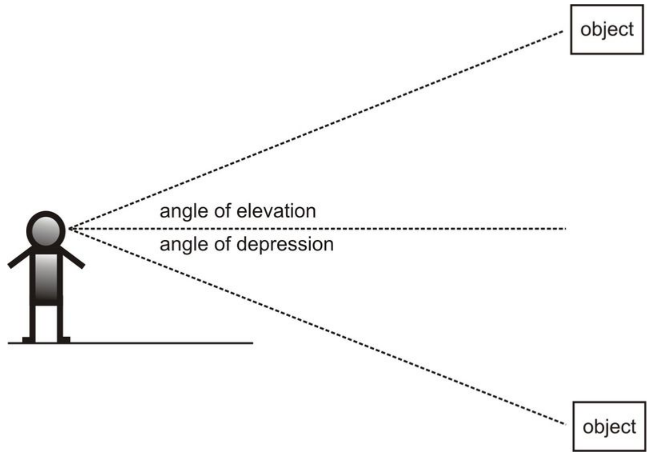


Solution: Recall that a parallelogram can be split into two triangles. So the formula for a parallelogram, using the new formula, would be: $A = 2 \cdot \frac{1}{2}\, ab \sin C$ or $A = ab \sin C$.

$$\begin{aligned} A &= 7 \cdot 15 \cdot \sin 65^\circ \\ &= 95.162 \end{aligned}$$

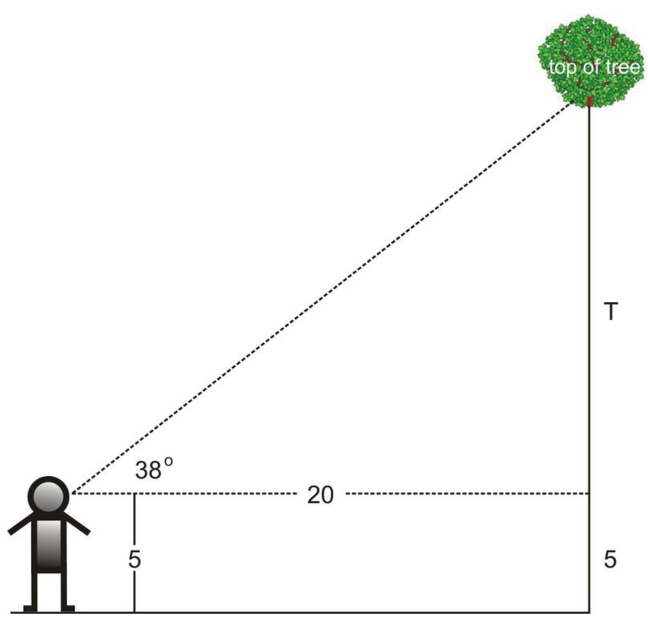
Angles of Elevation and Depression

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.



The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight *down to* an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Example 7: You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?



Solution:

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

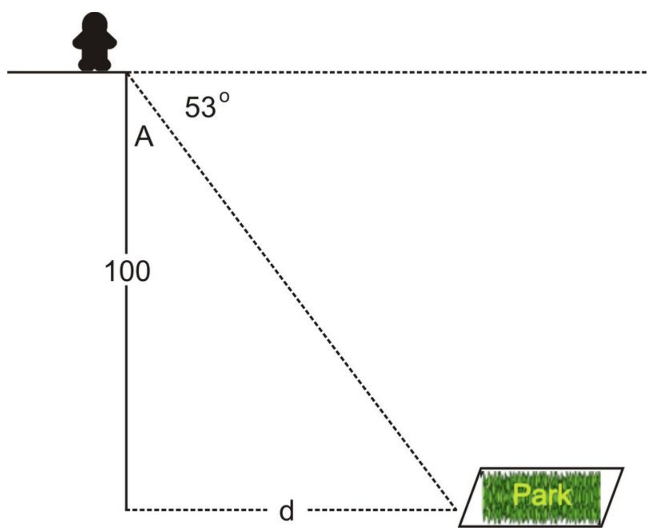
$$\tan 38^\circ = \frac{opposite}{adjacent} = \frac{T}{20}$$
$$\tan 38^\circ = \frac{T}{20}$$
$$T = 20\tan 38^\circ \approx 15.63$$
$$\text{Height of tree} \approx 20.63\ ft$$

The next example shows an angle of depression.

Example 8: You are standing on top of a building, looking at a park in the distance. The angle of depression is 53°. If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Solution:

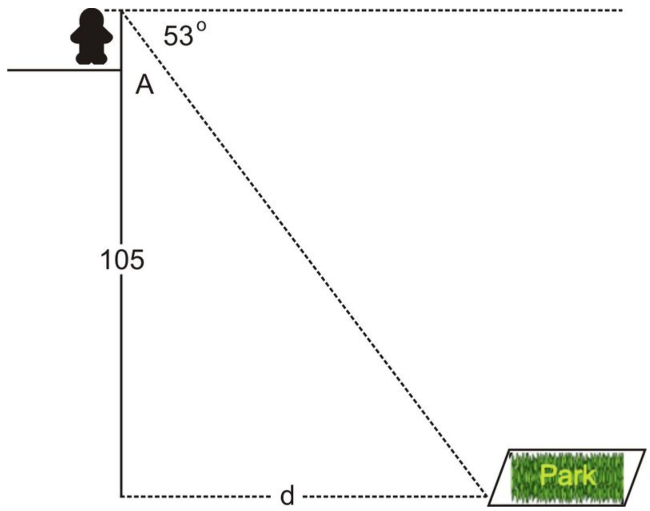
If we ignore the height of the person, we solve the following triangle:



Given the angle of depression is 53°, $\angle A$ in the figure above is 37°. We can use the tangent function to find the distance from the building to the park:

$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{100}$$
$$\tan 37^\circ = \frac{d}{100}$$
$$d = 100\tan 37^\circ \approx 75.36\ ft.$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

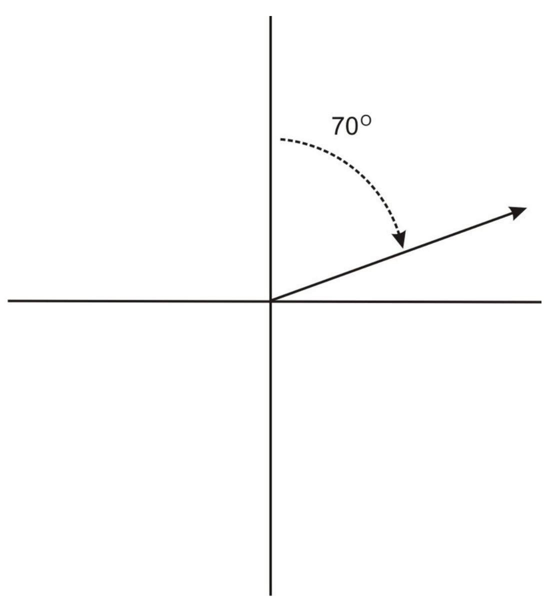


$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{105}$$
$$\tan 37^\circ = \frac{d}{105}$$
$$d = 105 \tan 37^\circ \approx 79.12\ ft.$$

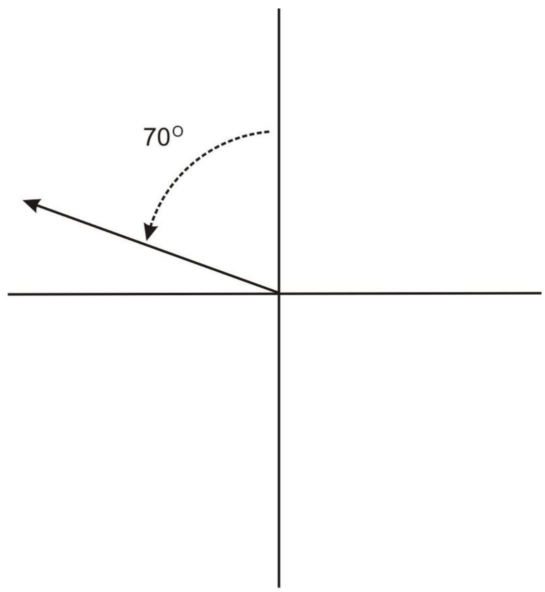
If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Right Triangles and Bearings

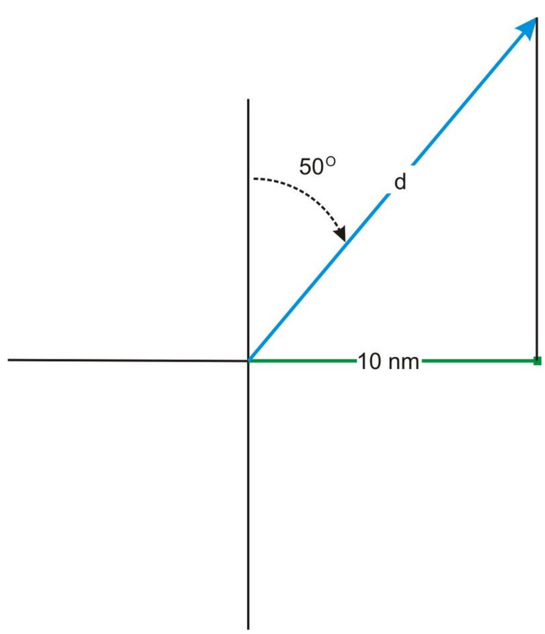
We can also use right triangles to find distances using angles given as bearings. In navigation, a bearing is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north. The graph below shows an angle of 70 degrees:



It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are measured in trigonometry. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70^\circ E$ refers to an angle from the north, towards the east, while $N70^\circ W$ refers to an angle from the north, towards the west. $N70^\circ E$ is the same as the angle shown in the graph above. $N70^\circ W$ would result in an angle in the second quadrant.



Example 9: A ship travels on a $N50^\circ E$ course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?



Solution: The angle between d and 10 nm is the complement of 50°, which is 40°. Therefore we can find d using the cosine function:

$$\cos 40^\circ = \frac{adjacent}{hypotenuse} = \frac{10}{d}$$
$$\cos 40^\circ = \frac{10}{d}$$
$$d \cos 40^\circ = 10$$
$$d = \frac{10}{\cos 40^\circ} \approx 13.05\ nm$$

Other Applications of Right Triangles

In general, you can use trigonometry to solve any problem that involves right triangles. The next few examples show different situations in which a right triangle can be used to find a length or a distance.

Example 10: The wheelchair ramp

In lesson 3 we introduced the following situation: You are building a ramp so that people in wheelchairs can access a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

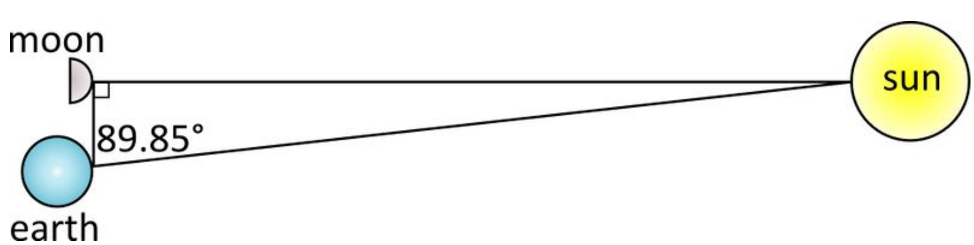
Given that we know the angle of the ramp and the length of the side opposite the angle, we can use the sine ratio to find the length of the ramp, which is the hypotenuse of the triangle:

$$\sin 5^\circ = \frac{8}{L}$$
$$L \sin 5^\circ = 8$$
$$L = \frac{8}{\sin 5^\circ} \approx 91.8\ ft$$

This may seem like a long ramp, but in fact a 5° ramp angle is what is required by the Americans with Disabilities Act (ADA). This explains why many ramps are comprised of several sections, or have turns. The additional distance is needed to make up for the small slope.

Right triangle trigonometry is also used for measuring distances that could not actually be measured. The next example shows a calculation of the distance between the moon and the sun. This calculation requires that we know the distance from the earth to the moon. In chapter 5 you will learn the Law of Sines, an equation that is necessary for the calculation of the distance from the earth to the moon. In the following example, we assume this distance, and use a right triangle to find the distance between the moon and the sun.

Example 11: The earth, moon, and sun create a right triangle during the first quarter moon. The distance from the earth to the moon is about 240,002.5 miles. What is the distance between the sun and the moon?



Solution:

Let $d =$ the distance between the sun and the moon. We can use the tangent function to find the value of d:

$$\tan 89.85^\circ = \frac{d}{240,002.5}$$
$$d = 240,002.5 \tan 89.85^\circ = 91,673,992.71 \textit{ miles}$$

Therefore the distance between the sun and the moon is much larger than the distance between the earth and the moon.

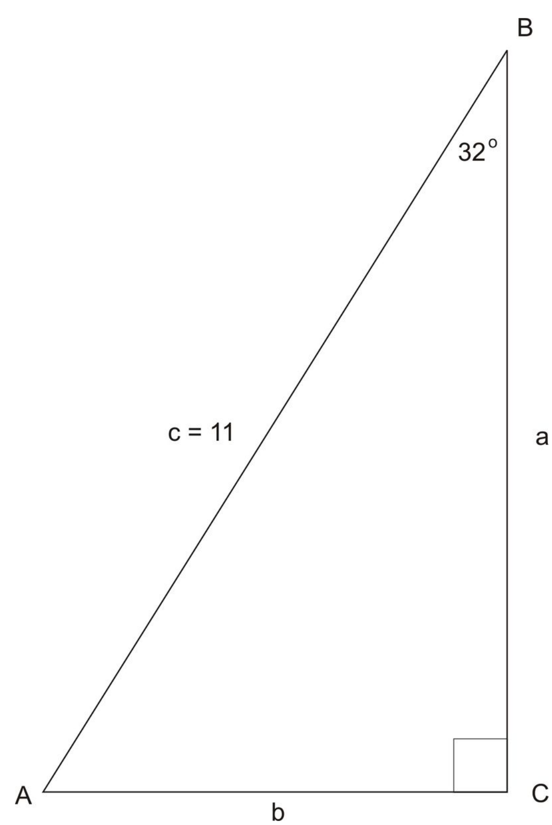
(Source: www.scribd.com, Trigonometry from the Earth to the Stars.)

Points to Consider

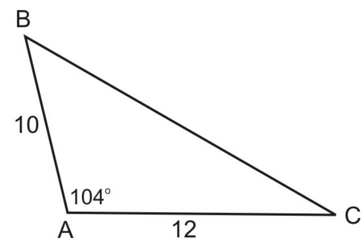
- In what kinds of situations do right triangles naturally arise?
- Are there right triangles that cannot be solved?
- Trigonometry can solve problems at an astronomical scale as well as problems at a molecular or atomic scale. Why is this true?

Review Questions

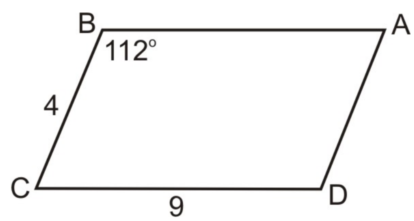
1. Solve the triangle.



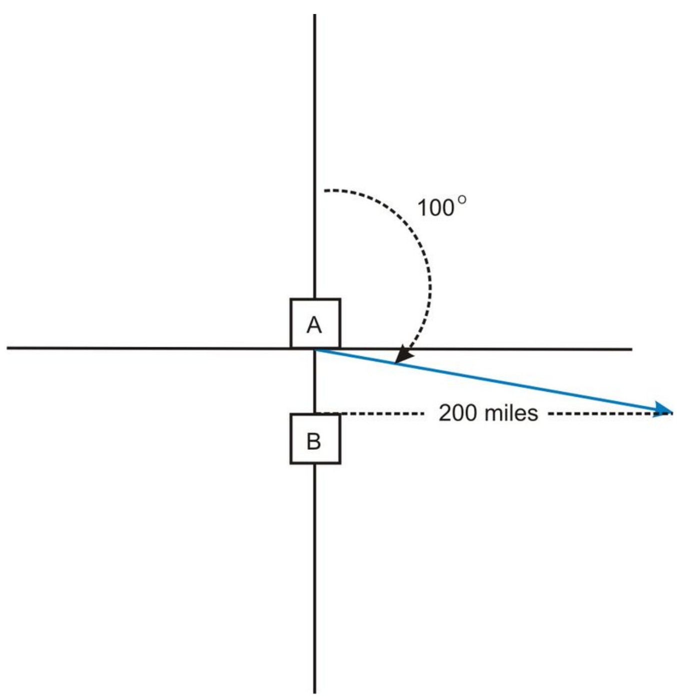
2. Two friends are writing practice problems to study for a trigonometry test. Sam writes the following problem for his friend Anna to solve: In right triangle ABC, the measure of angle C is 90 degrees, and the length of side c is 8 inches. Solve the triangle. Anna tells Sam that the triangle cannot be solved. Sam says that she is wrong. Who is right? Explain your thinking.
3. Use the Pythagorean Theorem to verify the sides of the triangle in example 2.
4. Estimate the measure of angle B in the triangle below using the fact that $\sin B = \frac{3}{5}$ and $\sin 30^\circ = \frac{1}{2}$. Use a calculator to find sine values. Estimate B to the nearest degree.
5. Find the area of the triangle.



6. Find the area of the parallelogram below.

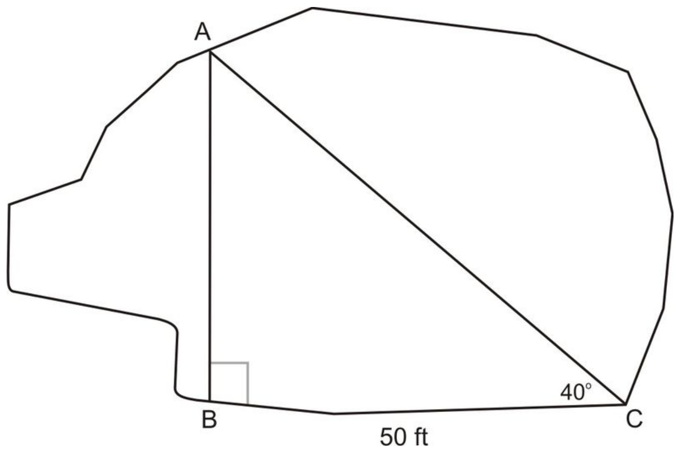


7. The angle of elevation from the ground to the top of a flagpole is measured to be 53°. If the measurement was taken from 15 feet away, how tall is the flagpole?
8. From the top of a hill, the angle of depression to a house is measured to be 14°. If the hill is 30 feet tall, how far away is the house?
9. An airplane departs city A and travels at a bearing of 100°. City B is directly south of city A. When the plane is 200 miles east of city B, how far has the plan traveled? How far apart are City A and City B?

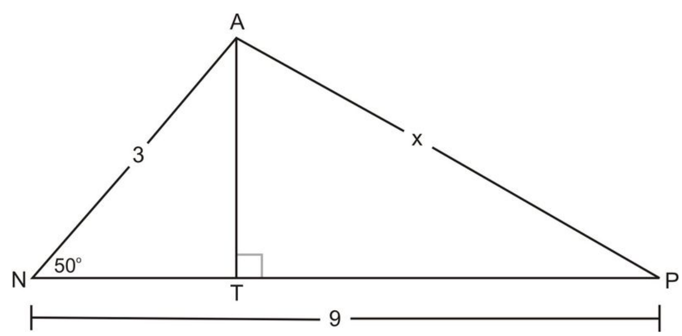


What is the length of the slanted outer wall, w? What is the length of the main floor, f?

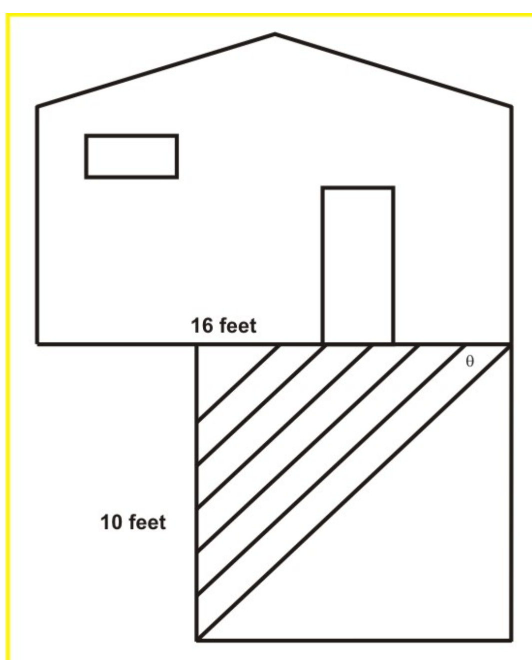
10. A surveyor is measuring the width of a pond. She chooses a landmark on the opposite side of the pond, and measures the angle to this landmark from a point 50 feet away from the original point. How wide is the pond?



11. Find the length of side x:



12. A deck measuring 10 feet by 16 feet will require laying boards with one board running along the diagonal and the remaining boards running parallel to that board. The boards meeting the side of the house must be cut prior to being nailed down. At what angle should the boards be cut?

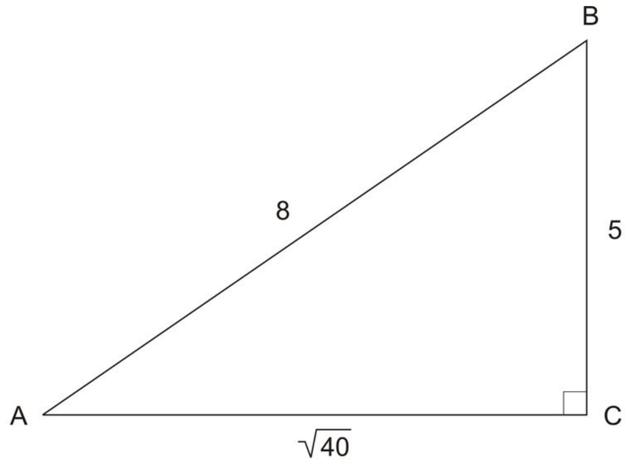


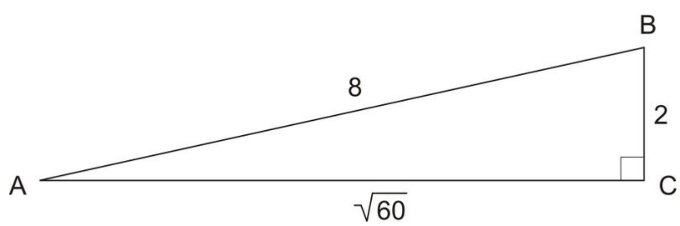
Review Answers

1.

$$\angle A = 50^\circ$$
$$b \approx 5.83$$
$$a \approx 9.33$$

2. Anna is correct. There is not enough information to solve the triangle. That is, there are infinitely many right triangles with hypotenuse 8. For example:





3. $6^2 + 5.03^2 = 36 + 25.3009 = 61.3009 = 7.83^2$.
4. $\angle B \approx 37^\circ$
5. $A = \frac{1}{2} \cdot 10 \cdot 12 \cdot \sin 104^\circ = 58.218$
6. $A = 4 \cdot 9 \cdot \sin 112^\circ = 33.379$
7. About 19.9 feet tall
8. About 120.3 feet
9. The plane has traveled about 203 miles. The two cities are 35 miles apart.
10. About 41.95 feet
11. About 7.44
12.

$$\tan\theta = \frac{opposite}{adjacent}$$
$$\tan\theta = 0.625$$
$$\theta = 32^\circ$$

1.5 Measuring Rotation

Learning Objectives

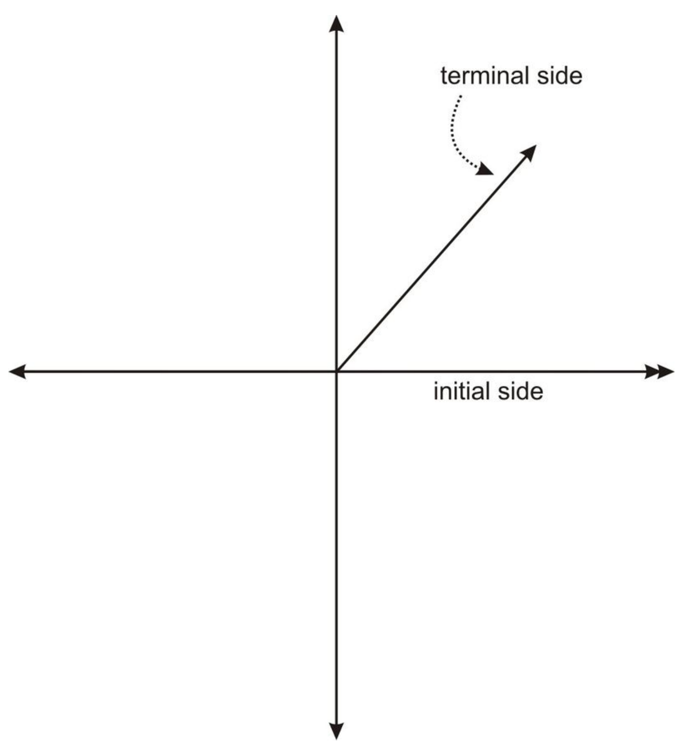
- Identify and draw angles of rotation in standard position.
- Identify quadrantal angles.
- Identify co-terminal angles.

Angles of Rotation in Standard Position

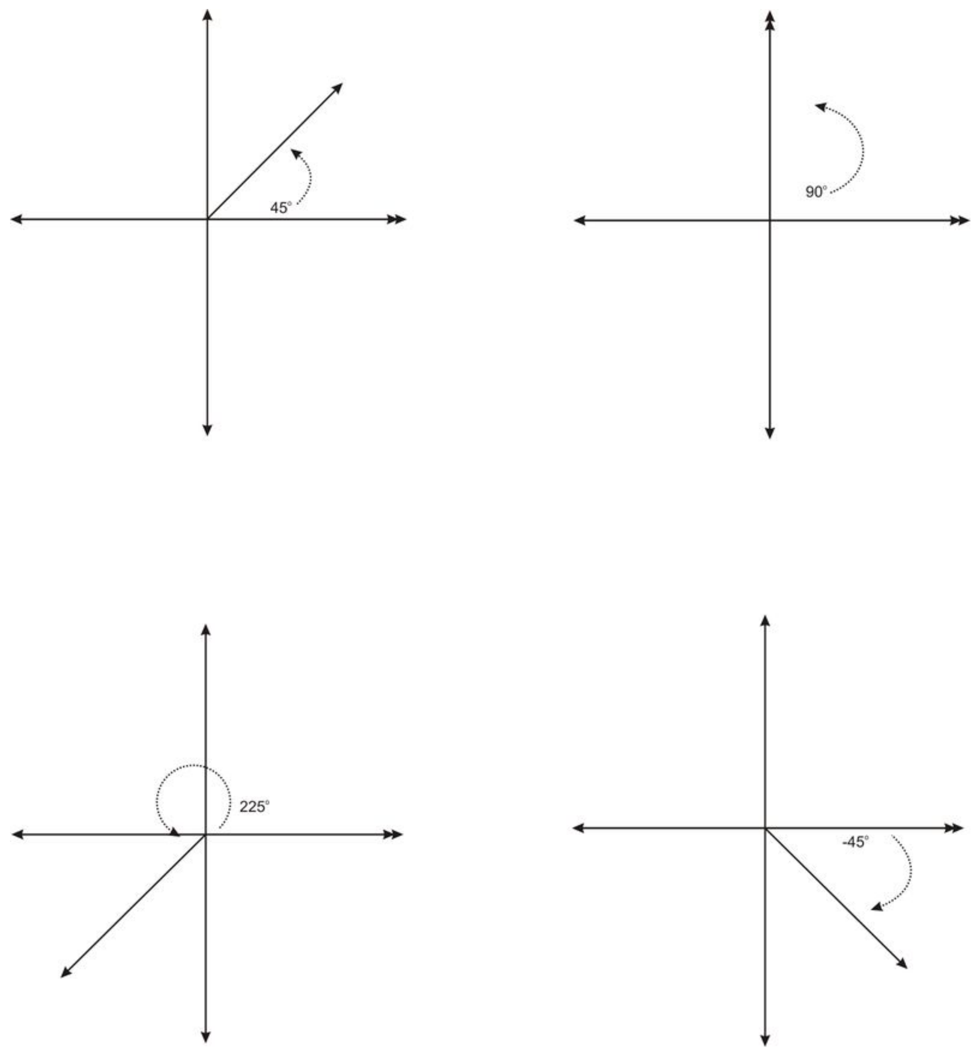
Consider, for example, a game that is played with a spinner. When you spin the spinner, how far has it gone? You can answer this question in several ways. You could say something like "the spinner spun around 3 times." This means that the spinner made 3 complete rotations, and then landed back where it started.

We can also measure the rotation in degrees. In the previous lesson we worked with angles in triangles, measured in degrees. You may recall from geometry that a full rotation is 360 degrees, usually written as 360°. Half a rotation is then 180° and a quarter rotation is 90°. Each of these measurements will be important in this lesson, as well as in the remainder of the chapter.

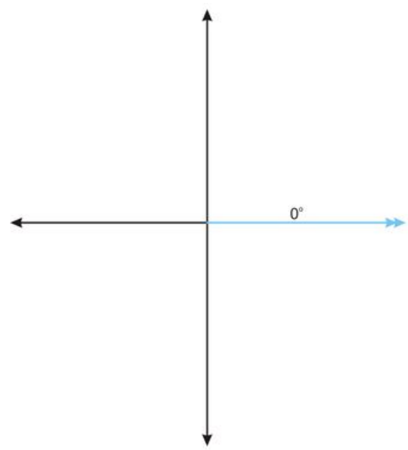
We can use our knowledge of graphing to represent any angle. The figure below shows an angle in what is called **standard position**.

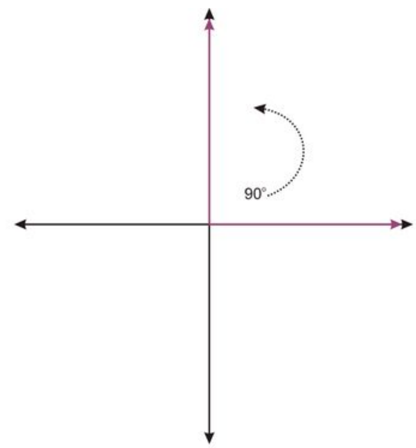


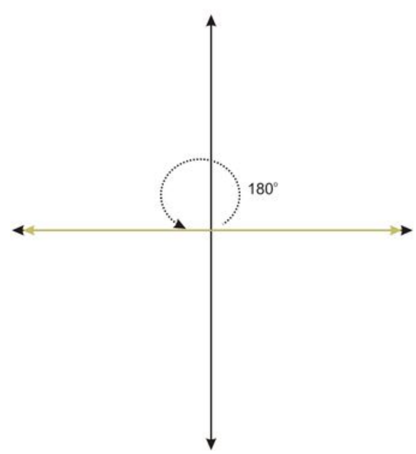
The initial side of an angle in standard position is always on the positive $x-$axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a **counterclockwise** direction. This means that if we rotate **clockwise**, we will generate a negative angle. Below are several examples of angles in standard position.

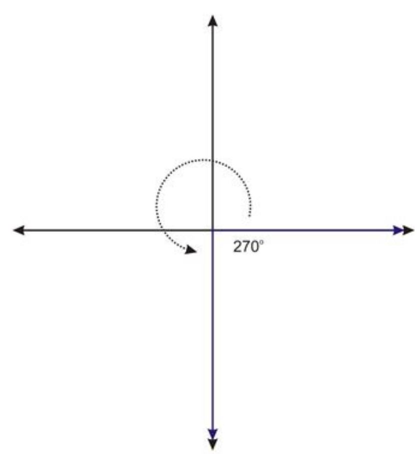


The 90 degree angle is one of four **quadrantal** angles. A quadrantal angle is one whose terminal side lies on an axis. Along with 90°, 0°, 180° and 270° are quadrantal angles.





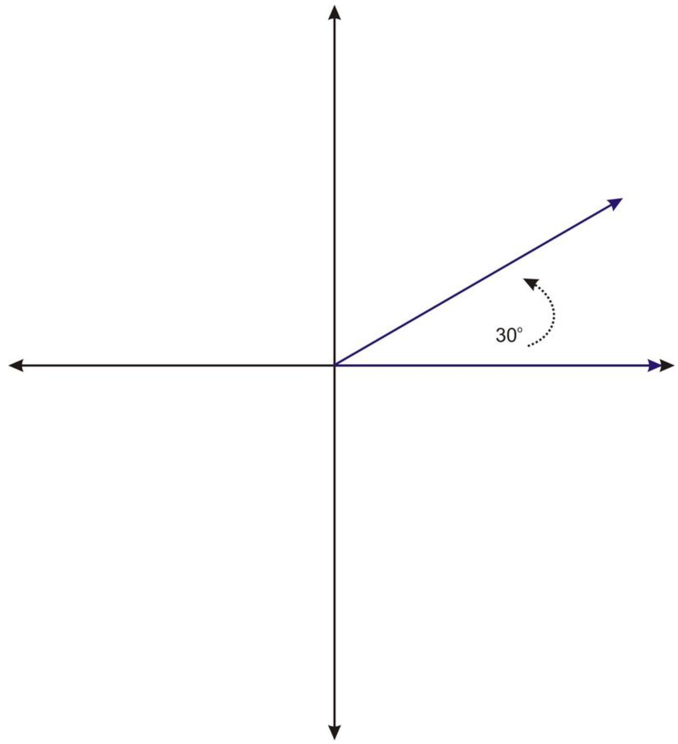




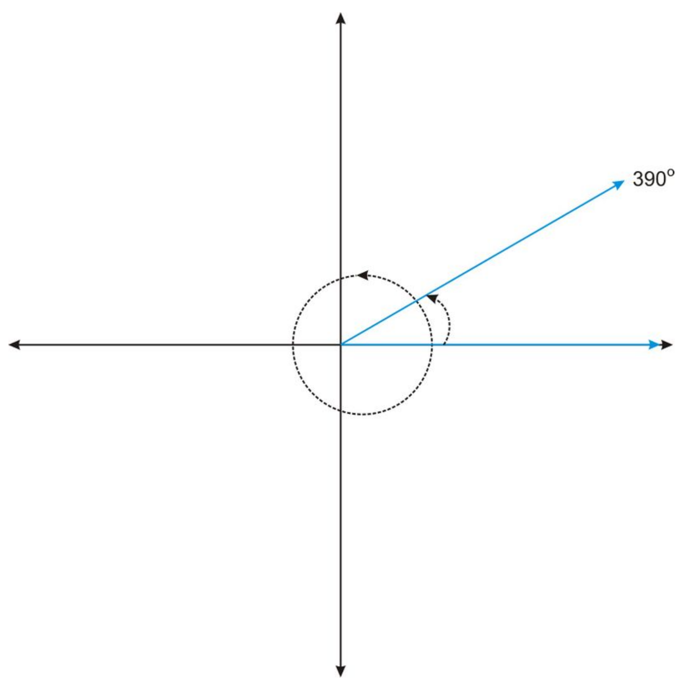
These angles are referred to as quadrantal because each angle defines a quadrant. Notice that without the arrow indicating the rotation, 270° looks as if it is a -90°, defining the fourth quadrant. Notice also that 360° would look just like 0°. The difference is in the action of rotation. This idea of two angles actually being the same angle is discussed next.

Coterminal Angles

Consider the angle 30°, in standard position.



Now consider the angle 390°. We can think of this angle as a full rotation (360°), plus an additional 30 degrees.



Notice that 390° looks the same as 30°. Formally, we say that the angles share the same terminal side. Therefore we call the angles **co-terminal**. Not only are these two angles co-terminal, but there are infinitely many angles that are co-terminal with these two angles. For example, if we rotate another 360°, we get the angle 750°. Or, if we create the angle in the negative direction (clockwise), we get the angle $-330°$. Because we can rotate in either direction, and we can rotate as many times as we want, we can continuously generate angles that are co-terminal with 30°.

Example 1: Which angles are co-terminal with 45°?

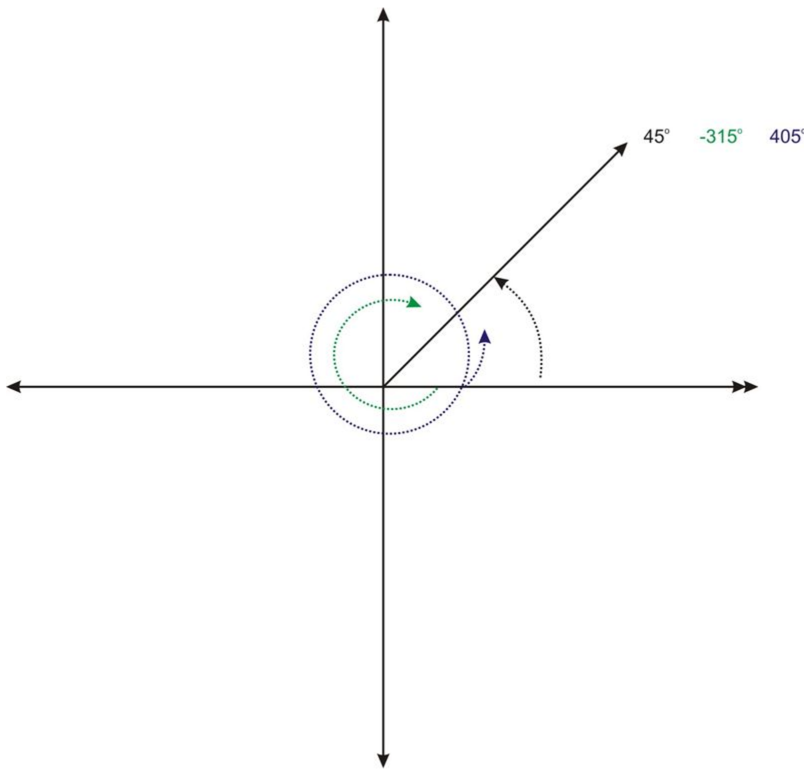
a. $-45°$

b. 405°

c. -315°

d. 135°

Solution: b. 405° and c. -315° are co-terminal with 45°.



Notice that terminal side of the first angle, -45°, is in the 4^{th} quadrant. The last angle, 135° is in the 2^{nd} quadrant. Therefore neither angle is co-terminal with 45°.

Now consider 405°. This is a full rotation, plus an additional 45 degrees. So this angle is co-terminal with 45°. The angle -315° can be generated by rotating clockwise. To determine where the terminal side is, it can be helpful to use quadrantal angles as markers. For example, if you rotate clockwise 90 degrees 3 times (for a total of 270 degrees), the terminal side of the angle is on the positive $y-$axis. For a total clockwise rotation of 315 degrees, we have $315 - 270 = 45$ degrees more to rotate. This puts the terminal side of the angle at the same position as 45°.

Points to Consider

- How can one angle look exactly the same as another angle?
- Where might you see angles of rotation in real life?

Review Questions

1. Plot the following angles in standard position.
   a. 60°
   b. -170°

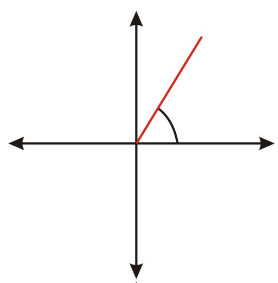
c. 365°
d. 325°
e. 240°

2. State the measure of an angle that is co-terminal with 90°.
3. Name a positive and negative angle that are co-terminal with:

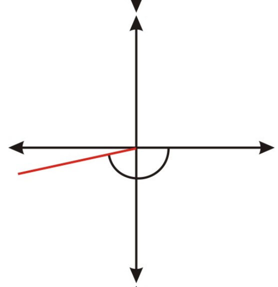
a. 120°
b. 315°
c. −150°

4. A drag racer goes around a 180 degree circular curve in a racetrack in a path of radius 120 m. Its front and back wheels have different diameters. The front wheels are 0.6 m in diameter. The rear wheels are much larger; they have a diameter of 1.8 m. The axles of both wheels are 2 m long. Which wheel has more rotations going around the curve? How many more degrees does the front wheel rotate compared to the back wheel?

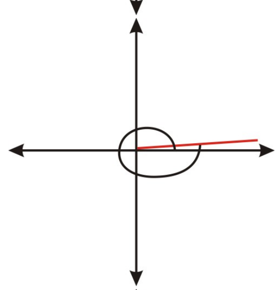
Review Answers



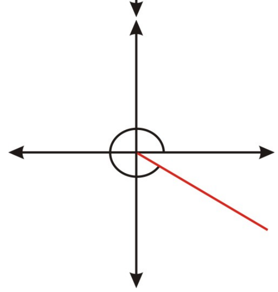
1. 1.



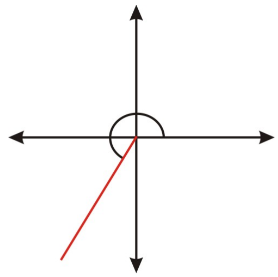
2.



3.



4.

5. 

2. Answers will vary. Examples: $450^\circ, -270^\circ$
   1. Answers will vary. Examples: $-240^\circ, 480^\circ$
   2. Answers will vary. Examples: $-45^\circ, 675^\circ$
   3. Answers will vary. Examples: $210^\circ, -510^\circ, 570^\circ$
3. The front wheel rotates more because it has a smaller diameter. It rotates 200 revolutions versus 66.67 revolutions for the back wheel, which is a $48,000^\circ$ difference $((200-66.\bar{6})\cdot 360^\circ)$.

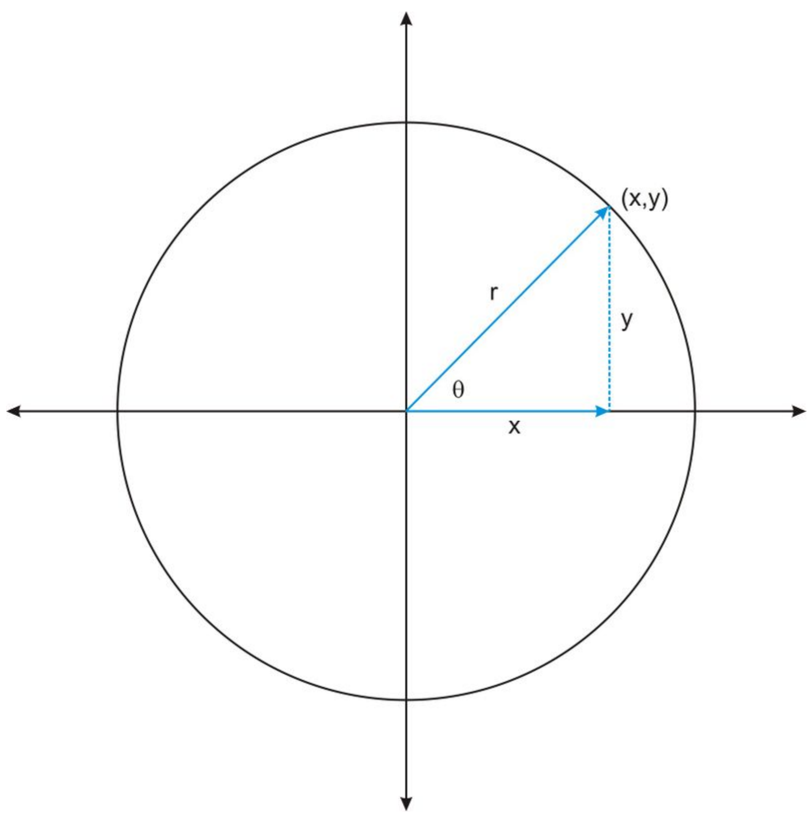
1.6 Applying Trig Functions to Angles of Rotation

Learning Objectives

- Find the values of the six trigonometric functions for angles of rotation.
- Recognize angles of the unit circle.

Trigonometric Functions of Angles in Standard Position

In section 1.3, we defined the six trigonometric functions for angles in right triangles. We can also define the same functions in terms of angles of rotation. Consider an angle in standard position, whose terminal side intersects a circle of radius r. We can think of the radius as the hypotenuse of a right triangle:



The point (x,y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of x, y, and r:

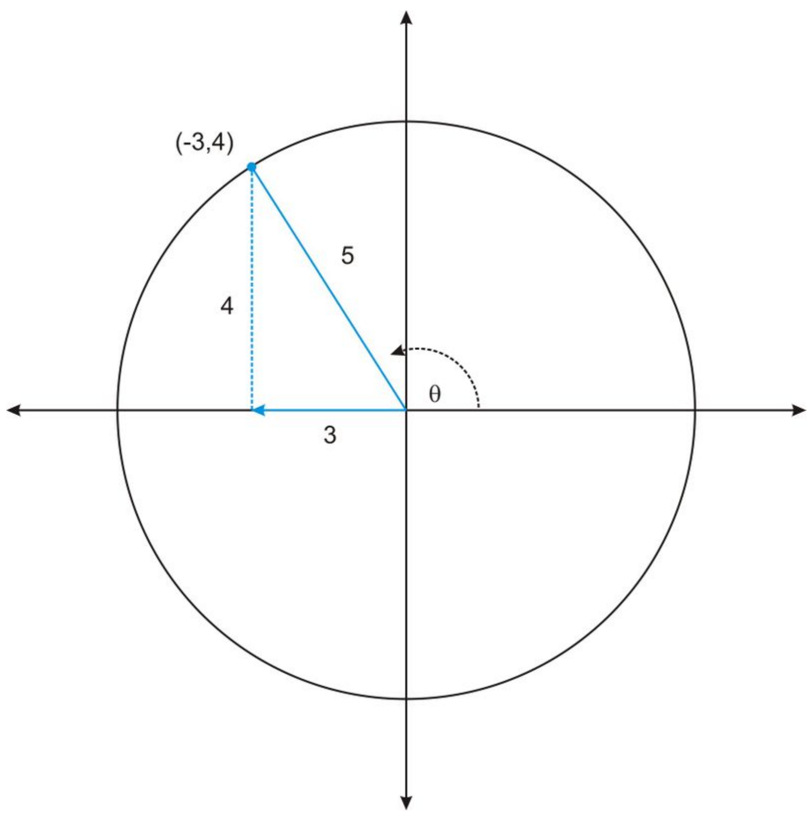
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

And, we can extend these functions to include non-acute angles.

Example 1: The point (-3, 4) is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Solution:

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

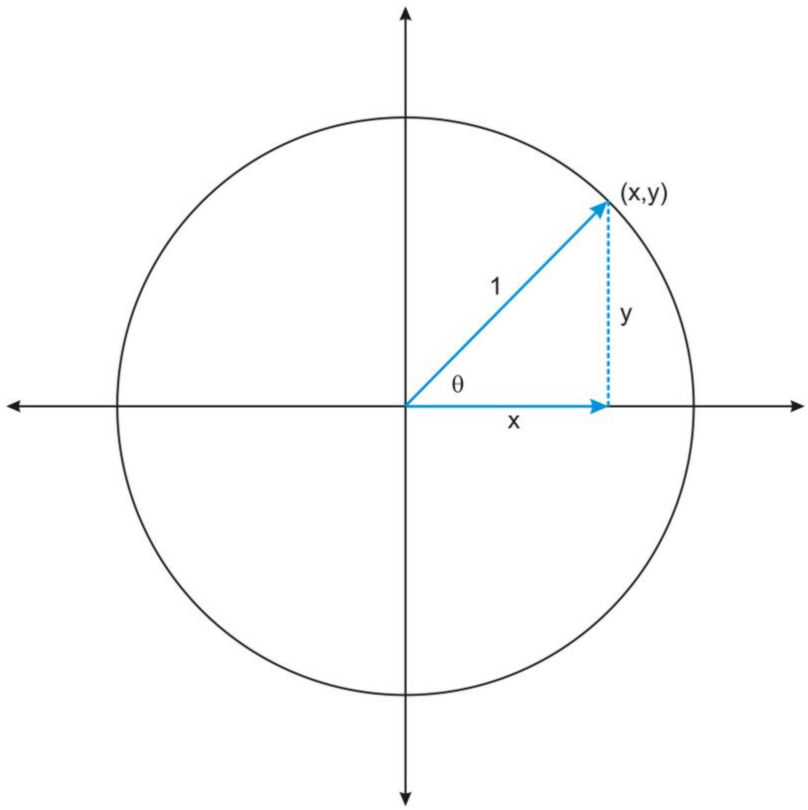


$$\cos\theta = \frac{-3}{5} \qquad \sec\theta = \frac{5}{-3}$$
$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$
$$\tan\theta = \frac{4}{-3} \qquad \cot\theta = \frac{-3}{4}$$

Notice that the value of r depends on the coordinates of the given point. You can always find the value of r using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you will see next, doing this allows us to simplify the definitions of the trig functions.

The Unit Circle

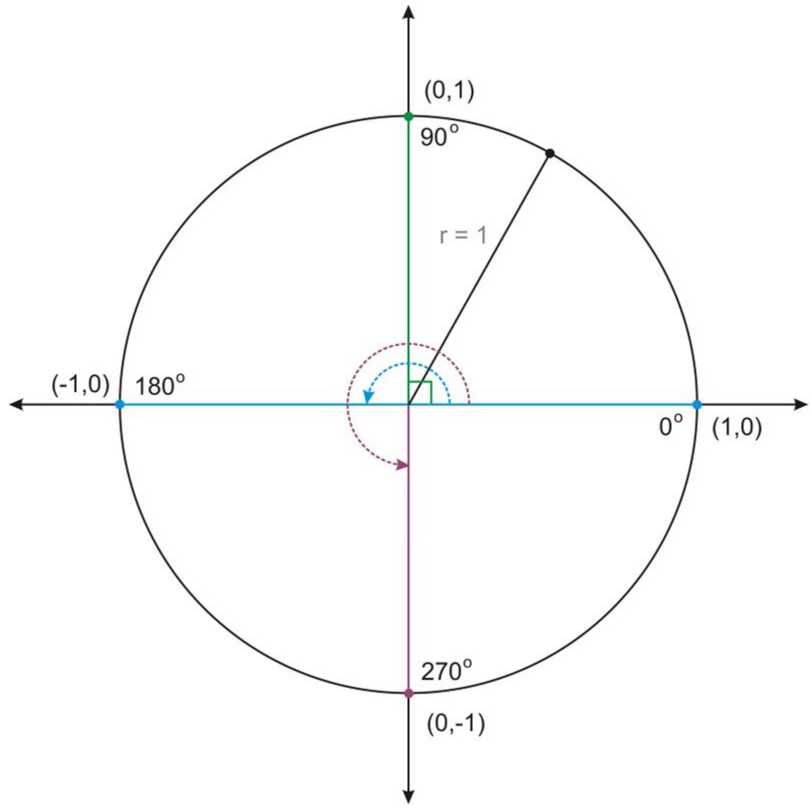
Consider an angle in standard position, such that the point (x,y) on the terminal side of the angle is a point on a circle with radius 1.



This circle is called the **unit circle**. With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the x and y coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.



We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $\sin 90^\circ = y = 1$.

Example 2: Use the unit circle above to find each value:

a. $\cos 90^\circ$

b. $\cot 180^\circ$

c. $\sec 0^\circ$

Solution:

a. $\cos 90^\circ = 0$

The ordered pair for this angle is (0, 1). The cosine value is the x coordinate, 0.

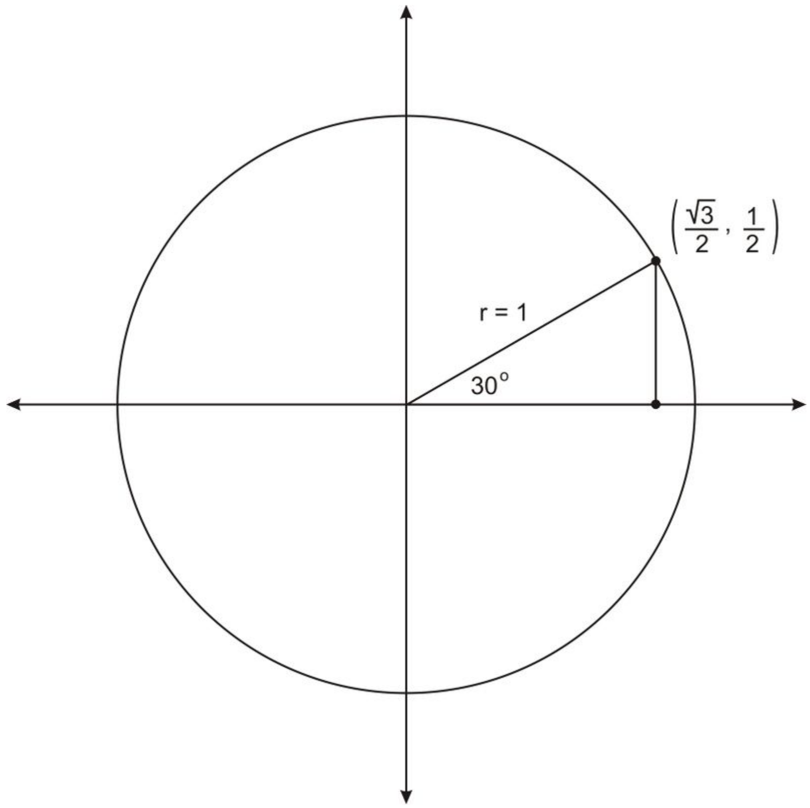
b. $\cot 180^\circ$ is undefined

The ordered pair for this angle is (-1, 0). The ratio $\frac{x}{y}$ is $\frac{-1}{0}$, which is undefined.

c. $\sec 0^\circ = 1$

The ordered pair for this angle is (1, 0). The ratio $\frac{1}{x}$ is $\frac{1}{1} = 1$.

There are several important angles in the unit circle that you will work with extensively in your study of trigonometry, primarily 30°, 45°, and 60°. Recall section 1.2 to find the values of the trigonometric functions of these angles. First, we need to know the ordered pairs. Let's begin with 30°.



This triangle is identical to #8 from 1.3. If you look back at this problem, you will recall that you found the sine, cosine and tangent of 30° and 60°. It is no coincidence that the endpoint on the unit circle is the same as your answer from #8.

The terminal side of the angle intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore we can find the values of any of the trig functions of 30°. For example, the cosine value is the $x-$coordinate, so $\cos(30^\circ) = \frac{\sqrt{3}}{2}$. Because the coordinates are fractions, we have to do a bit more work in order to find the tangent value:

$$\tan 30^\circ = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

In the review exercises you will find the values of the remaining four trig functions of this angle. The table below summarizes the ordered pairs for 30°, 45°, and 60° on the unit circle.

TABLE 1.1:

Angle	$x-$**coordinate**	$y-$**coordinate**
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

We can use these values to find the values of any of the six trig functions of these angles.

Example 3: Find the value of each function.

a. $\cos 45^\circ$

b. $\sin 60^\circ$

c. $\tan 45^\circ$

Solution:

a. $\cos 45^\circ = \frac{\sqrt{2}}{2}$ The cosine value is the $x-$ coordinate of the point.

b. $\sin 60^\circ = \frac{\sqrt{3}}{2}$ The sine value is the $y-$ coordinate of the point.

c. $\tan 45^\circ = 1$ The tangent value is the ratio of the $y-$ coordinate to the $x-$ coordinate. Because the $x-$ and $y-$ coordinates are the same for this angle, the tangent ratio is 1.

Points to Consider

- How can some values of the trig functions be negative? How is it that some are undefined?
- Why is the unit circle and the trig functions defined on it useful, even when the hypotenuses of triangles in the problem are not 1?

Review Questions

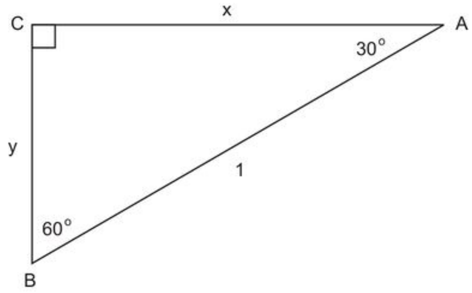
1. The point (3, -4) is a point on the terminal side of an angle θ in standard position.

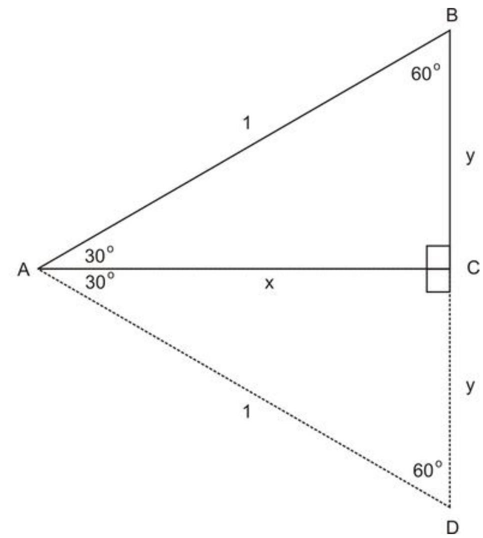
   a. Determine the radius of the circle.
   b. Determine the values of the six trigonometric functions of the angle.

2. The point (-5, -12) is a point on the terminal side of an angle θ in standard position.

   a. Determine the radius of the circle.
   b. Determine the values of the six trigonometric functions of the angle.

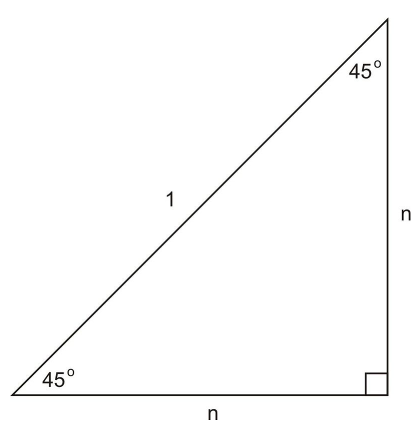
3. $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$. Find $\sin\theta$.
4. $\csc\theta = -4$ and $\tan\theta > 0$. Find the exact values of the remaining trigonometric functions.
5. (2, 6) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.
6. The terminal side of the angle 270° intersects the unit circle at (0, -1). Use this ordered pair to find the six trig functions of 270°.
7. In the lesson you learned that the terminal side of the angle 30° intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Here you will prove that this is true.





a. Explain why Triangle ABD is an equiangular triangle. What is the measure of angle DAB?
b. What is the length of BD? How do you know?
c. What is the length of BC and CD? How do you know?
d. Now explain why the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
e. Why does this tell you that the ordered pair for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$?

8. In the lesson you learned that the terminal side of the angle 45° is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Use the figure below and the Pythagorean Theorem to show that this is true.



9. In what quadrants will an angle in standard position have a positive tangent value? Explain your thinking.
10. Sketch the angle 150° on the unit circle is. How is this angle related to 30°? What do you think the ordered pair is?
11. We now know that $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \frac{y}{x}$. First, explain how it looks as though sine, cosine, and tangent are related. Second, can you rewrite tangent in terms of sine and cosine?

Review Answers

1. The radius of the circle is 5.

$$\cos\theta = \frac{3}{5} \qquad \sec\theta = \frac{5}{3}$$
$$\sin\theta = \frac{-4}{5} \qquad \csc\theta = \frac{5}{-4}$$
$$\tan\theta = \frac{-4}{3} \qquad \cot\theta = \frac{3}{-4}$$

2. The radius of the circle is 13.

$$\cos\theta = \frac{-5}{13} \qquad \sec\theta = \frac{13}{-5}$$
$$\sin\theta = \frac{-12}{13} \qquad \csc\theta = \frac{13}{-12}$$
$$\tan\theta = \frac{-12}{-5} = \frac{12}{5} \qquad \cot\theta = \frac{-5}{-12} = \frac{5}{12}$$

3. If $\tan\theta = -\frac{2}{3}$, it must be in either Quadrant II or IV. Because $\cos\theta > 0$, we can eliminate Quadrant II. So, this means that the 3 is negative. (All Students Take Calculus) From the Pythagorean Theorem, we find the hypotenuse:

$$2^2 + (-3^2) = c^2$$
$$4 + 9 = c^2$$
$$13 = c^2$$
$$\sqrt{13} = c$$

Because we are in Quadrant IV, the sine is negative. So, $\sin\theta = -\frac{2}{\sqrt{13}}$ or $-\frac{2\sqrt{13}}{13}$ (Rationalize the denominator)

4. If $\csc\theta = -4$, then $\sin\theta = -\frac{1}{4}$, sine is negative, so θ is in either Quadrant III or IV. Because $\tan\theta > 0$, we can eliminate Quadrant IV, therefore θ is in Quadrant III. From the Pythagorean Theorem, we can find the other leg:

$$a^2 + (-1)^2 = 4^2$$
$$a^2 + 1 = 16$$
$$a^2 = 15$$
$$a = \sqrt{15}$$
$$\text{So, } \cos\theta = -\frac{\sqrt{15}}{4}, \sec\theta = -\frac{4}{\sqrt{15}} \text{ or } -\frac{4\sqrt{15}}{15}$$
$$\tan\theta = -\frac{1}{\sqrt{15}} \text{ or } \frac{\sqrt{15}}{15}, \cot\theta = \sqrt{15}$$

5. If the terminal side of θ is on (2, 6) it means θ is in Quadrant I, so sine, cosine and tangent are all positive. From the Pythagorean Theorem, the hypotenuse is:

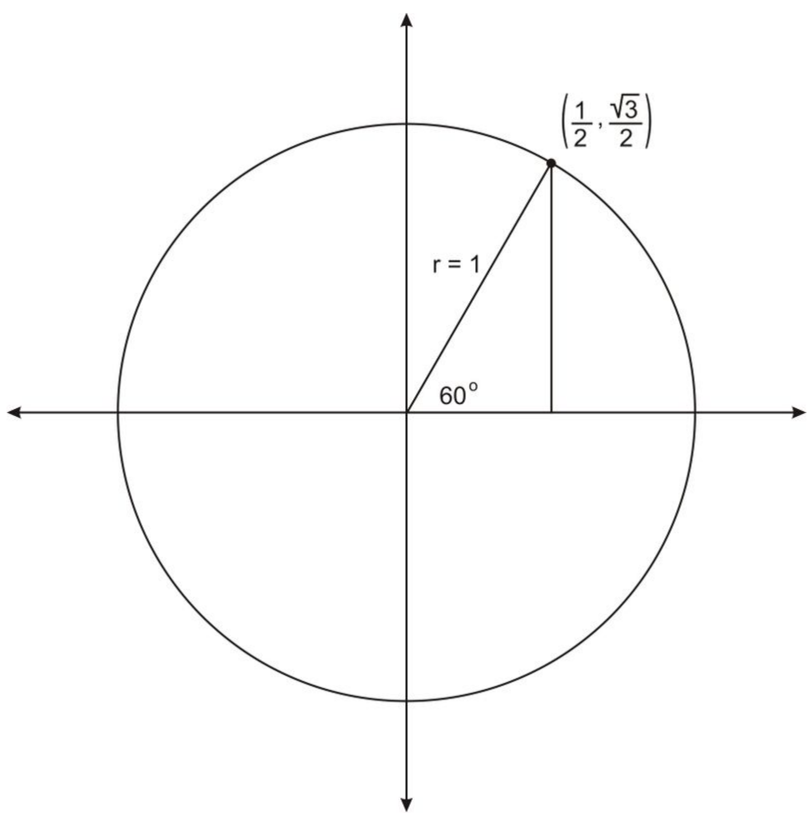
$$2^2 + 6^2 = c^2$$
$$4 + 36 = c^2$$
$$40 = c^2$$
$$\sqrt{40} = 2\sqrt{10} = c$$

Therefore, $\sin\theta = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}, \cos\theta = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$ and $\tan\theta = \frac{6}{2} = 3$

6.

$$\begin{aligned} \cos 270 &= 0 & \sec 270 &= undefined \\ \sin 270 &= -1 & \csc 270 &= \frac{1}{-1} = -1 \\ \tan 270 &= undefined & \cot 270 &= 0 \end{aligned}$$

1. The triangle is equiangular because all three angles measure 60 degrees. Angle *DAB* measures 60 degrees because it is the sum of two 30 degree angles.
2. *BD* has length 1 because it is one side of an equiangular, and hence equilateral, triangle.
3. *BC* and *CD* each have length $\frac{1}{2}$, as they are each half of *BD*. This is the case because Triangle *ABC* and *ADC* are congruent.
4. We can use the Pythagorean theorem to show that the length of *AC* is $\frac{\sqrt{3}}{2}$. If we place angle *BAC* as an angle in standard position, then *AC* and *BC* correspond to the *x* and *y* coordinates where the terminal side of the angle intersects the unit circle. Therefore the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
5. If we draw the angle 60° in standard position, we will also obtain a $30-60-90$ triangle, but the side lengths will be interchanged. So the ordered pair for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



7.

$$
\begin{aligned}
n^2+n^2 &= 1^2 \\
2n^2 &= 1 \\
n^2 &= \frac{1}{2} \\
n &= \pm\sqrt{\frac{1}{2}} \\
n &= \pm\frac{1}{\sqrt{2}} \\
n &= \pm\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}
\end{aligned}
$$

Because the angle is in the first quadrant, the x and y coordinates are positive.

8. An angle in the first quadrant, as the tangent is the ratio of two positive numbers. And, angle in the third quadrant, as the tangent in the ratio of two negative numbers, which will be positive.
9. The terminal side of the angle is a reflection of the terminal side of 30°. From this, students should see that the ordered pair is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
10. Students should notice that tangent is the ratio of $\frac{\sin}{\cos}$, which is $\frac{y}{x}$, which is also slope.

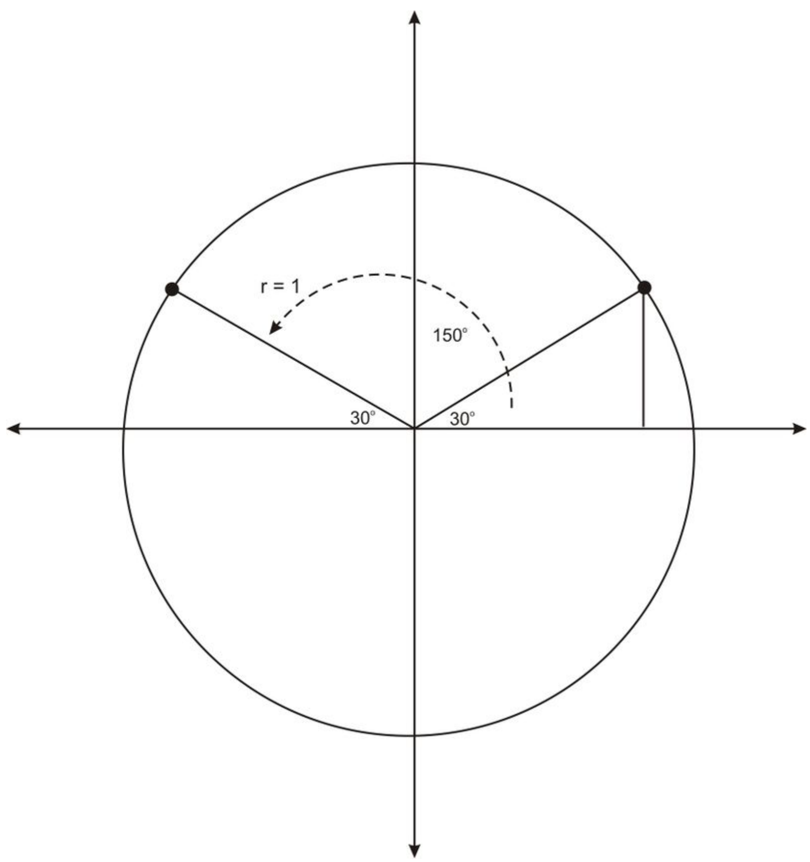
1.7 Trigonometric Functions of Any Angle

Learning Objectives

- Identify the reference angles for angles in the unit circle.
- Identify the ordered pair on the unit circle for angles whose reference angle is 30°, 45°, and 60°, or a quadrantal angle, including negative angles, and angles whose measure is greater than 360°.
- Use these ordered pairs to determine values of trig functions of these angles.
- Use calculators to find values of trig functions of any angle.

Reference Angles and Angles in the Unit Circle

In the previous lesson, one of the review questions asked you to consider the angle 150°. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of 30°, across the $y-$axis.



Notice that 150° makes a 30° angle with the negative $x-$axis. Therefore we say that 30° is the **reference angle** for 150°. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the $x-$axis. Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.

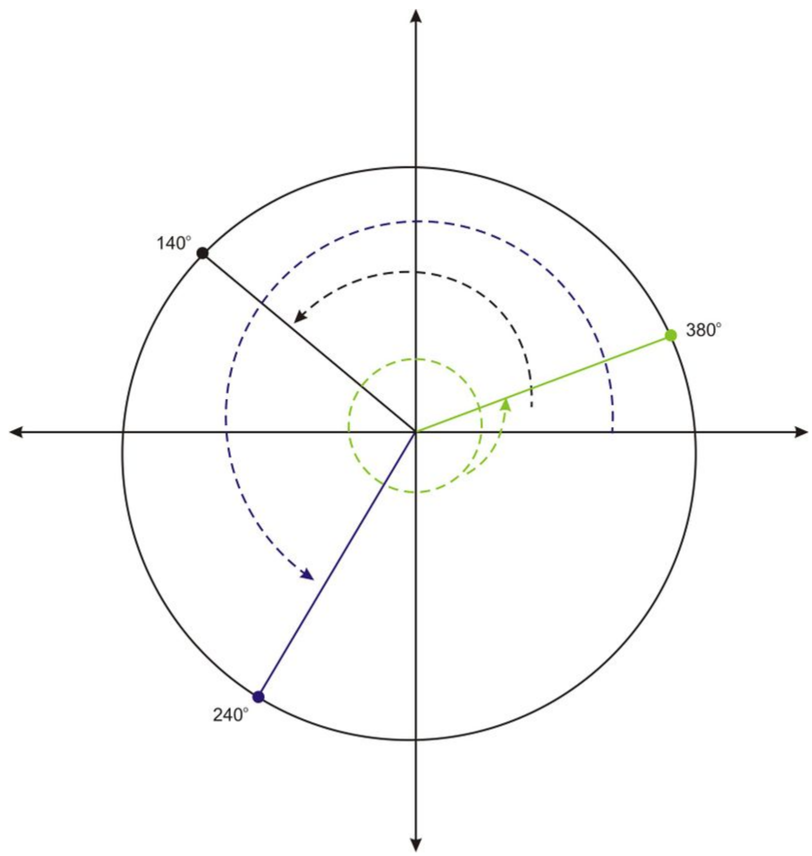
Example 1: Graph each angle and identify its reference angle.

a. 140°

b. 240°

c. 380°

Solution:



a. 140° makes a 40° angle with the $x-$axis. Therefore the reference angle is 40°.

b. 240° makes a 60° with the $x-$axis. Therefore the reference angle is 60°.

c. 380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

If an angle has a reference angle of 30°, 45°, or 60°, we can identify its ordered pair on the unit circle, and so we can find the values of the six trig functions of that angle. For example, above we stated that 150° has a reference angle of 30°. Because of its relationship to 30°, the ordered pair for is 150° is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Now we can find the values of the six trig functions of 150°:

$$\cos 150 = x = \frac{-\sqrt{3}}{2} \qquad \sec 150 = \frac{1}{x} = \frac{1}{\frac{-\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}}$$

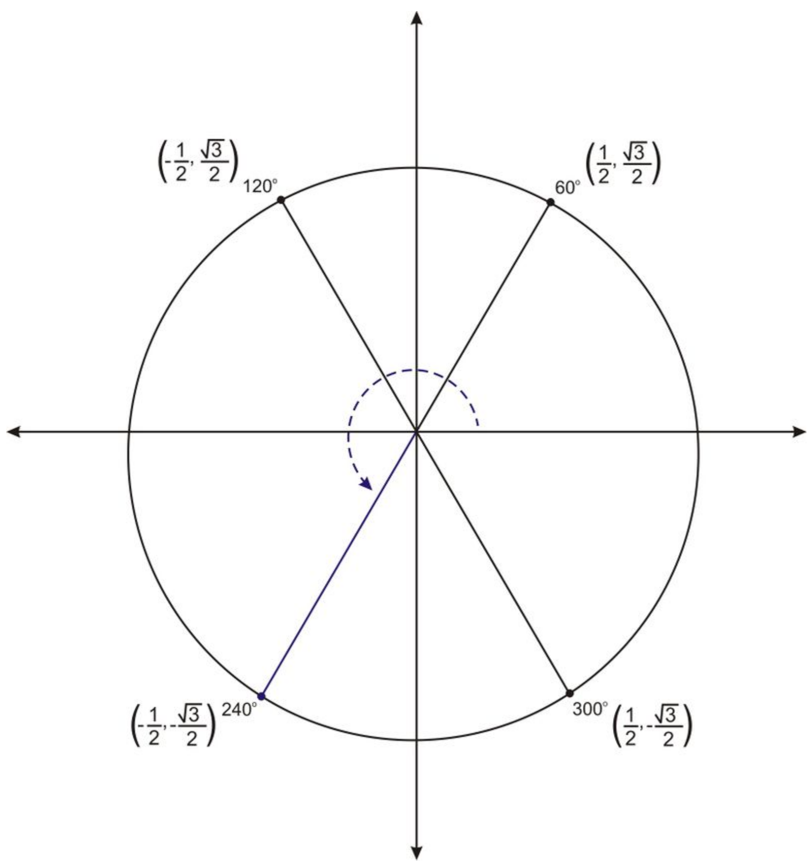
$$\sin 150 = y = \frac{1}{2} \qquad \csc 150 = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\tan 150 = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{-\sqrt{3}} \qquad \cot 150 = \frac{x}{y} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 2: Find the ordered pair for 240° and use it to find the value of sin 240°.

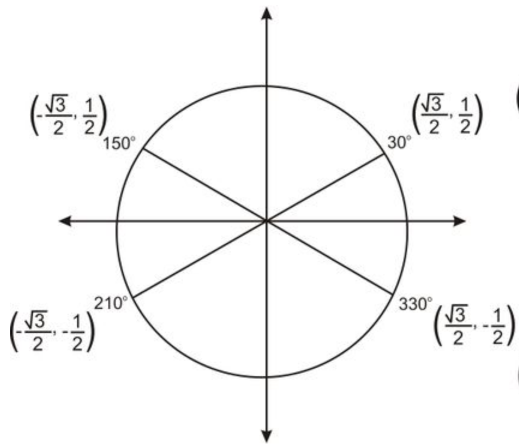
Solution: $\sin 240^\circ = \frac{-\sqrt{3}}{2}$

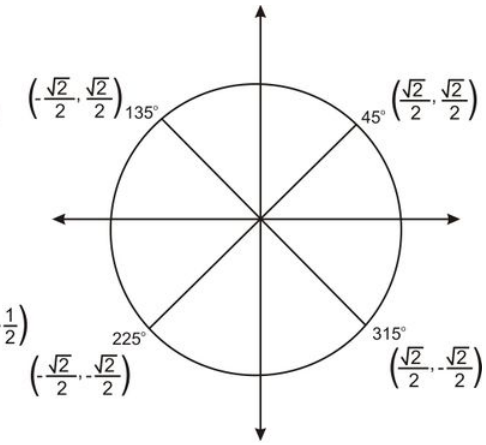
As we found in example 1, the reference angle for 240° is 60°. The figure below shows 60° and the three other angles in the unit circle that have 60° as a reference angle.



The terminal side of the angle 240° represents a reflection of the terminal side of 60° over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The $y-$coordinate is the sine value, so $\sin 240^\circ = \frac{-\sqrt{3}}{2}$.

Just as the figure above shows 60° and three related angles, we can make similar graphs for 30° and 45°.





Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

Example 3: Find the value of $\cot 300^\circ$

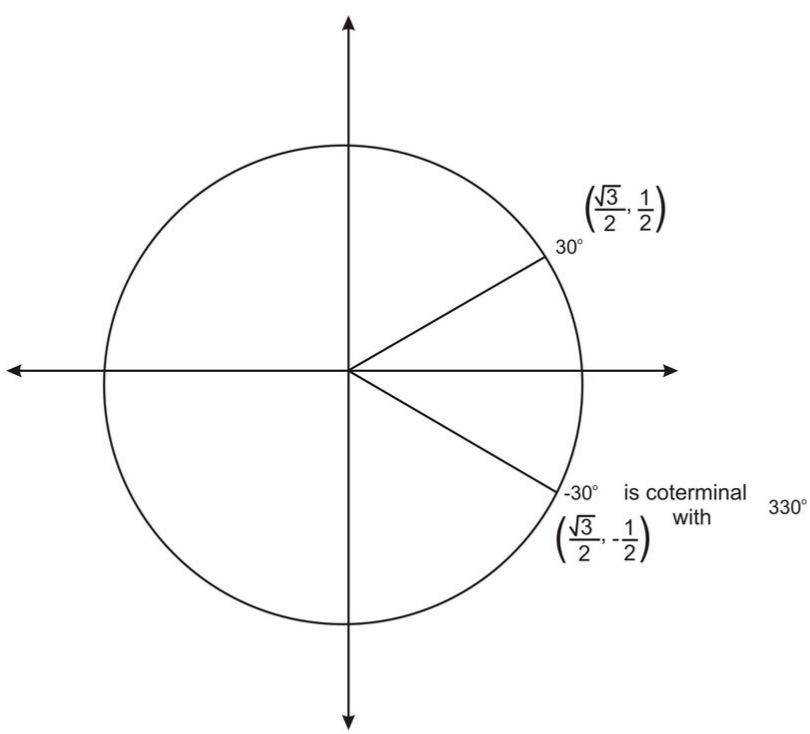
Solution: $\cot 300^\circ - \frac{1}{\sqrt{3}}$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is $\cot 300 = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows -30°.



Notice that this angle is coterminal with 330°. So the ordered pair is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. We can use this ordered pair to find the values of any of the trig functions of -30°. For example, $\cos(-30^\circ) = x = \frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.

Example 4: Find the value of each expression.

a. $\sin(-45^\circ)$

b. $\sec(-300^\circ)$

c. $\cos(-90^\circ)$

Solution:

a. $\sin(-45^\circ) = -\frac{\sqrt{2}}{2}$

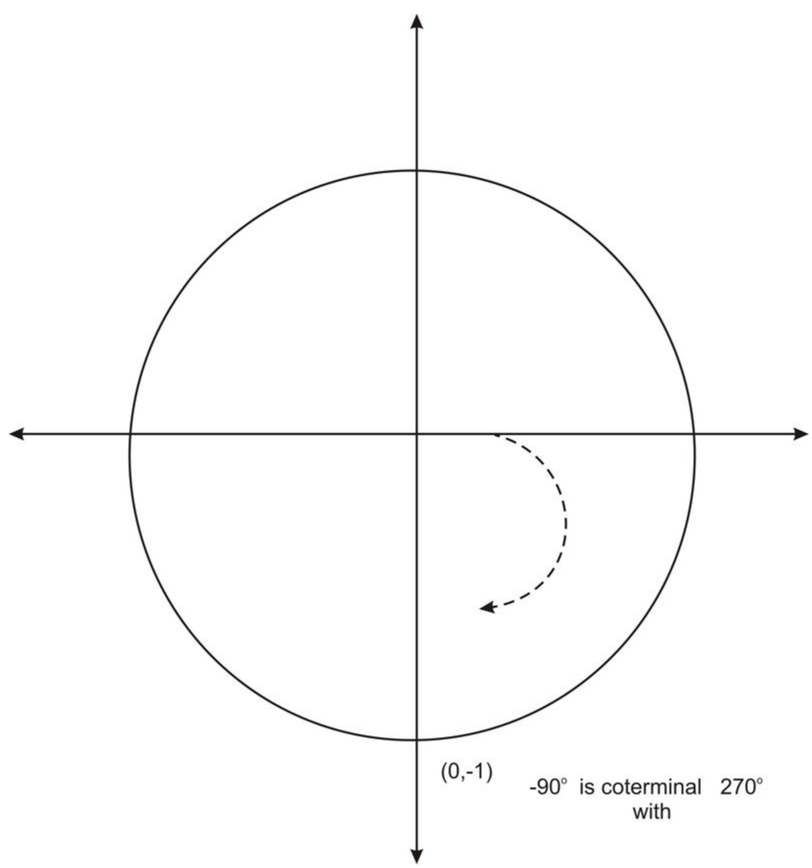
-45° is in the 4^{th} quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

b. $\sec(-300^\circ) = 2$

The angle -300° is in the 1^{st} quadrant and has a reference angle of 60°. That is, this angle is coterminal with 60°. Therefore the ordered pair is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the secant value is $\frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$.

c. $\cos(-90^\circ) = 0$

The angle -90° is coterminal with 270°. Therefore the ordered pair is (0, -1) and the cosine value is 0.

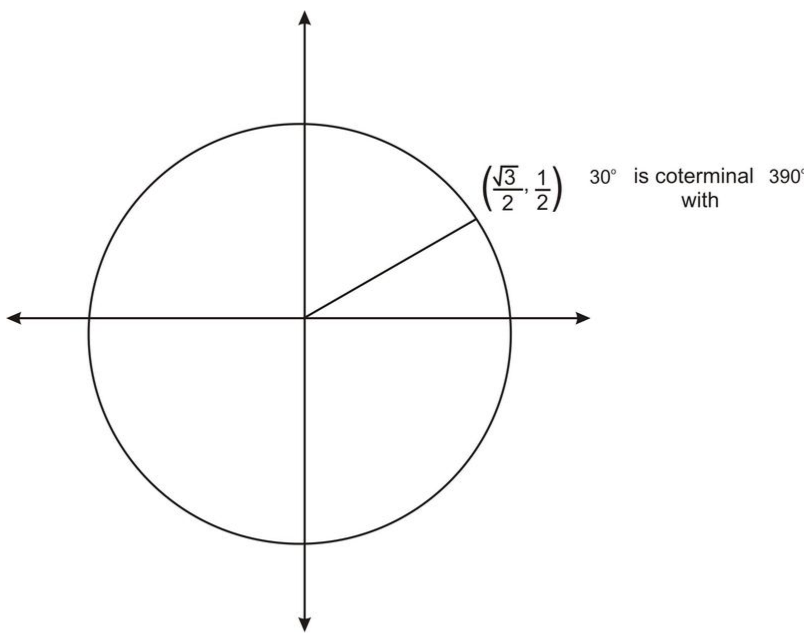


We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Trigonometric Functions of Angles Greater than 360 Degrees

Consider the angle 390°. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore 390° is coterminal with 30°. As you saw above with negative angles, this means that 390° has the same ordered pair as 30°, and so it has the same trig values. For example,

$$\cos 390^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$



In general, if an angle whose measure is greater than 360 has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. Again, determine the reference angle first.

Example 5: Find the value of each expression.

a. $\sin 420^\circ$

b. $\tan 840^\circ$

c. $\cos 540^\circ$

Solution:

a. $\sin 420^\circ = \frac{\sqrt{3}}{2}$

420° is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with 60°, and so it shares the same ordered pair, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The sine value is the $y-$coordinate.

b. $\tan 840^\circ = -\sqrt{3}$

840° is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore 840° is coterminal with 120°, so the ordered pair is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan 840^\circ = \tan 120^\circ = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

c. $\cos 540^\circ = -1$

540° is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with 180°, and the ordered pair is (-1, 0). So the cosine value is -1.

So far all of the angles we have worked with are multiples of 30, 45, 60, and 90. Next we will find approximate values of the trig functions of other angles.

Using a Calculator to Find Values

If you have a scientific calculator, you can determine the value of any trig function for any angle. Here we will focus on using a TI graphing calculator to find values.

First, your calculator needs to be in the correct "mode." In chapter 2 you will learn about a different system for measuring angles, known as radian measure. In this chapter, we are measuring angles in degrees. We need to make sure that the calculator is in degrees. To do this, press $\boxed{\text{MODE}}$. In the third row, make sure that Degree is highlighted. If Radian is highlighted, scroll down to this row, scroll over to Degree, and press $\boxed{\text{ENTER}}$. This will highlight Degree. Then press $\boxed{2^{\text{nd}}}\boxed{\text{MODE}}$ to return to the main screen.

Now you can calculate any value. For example, we can verify the values from the table above. To find $\sin 130^\circ$, press $\boxed{\text{Sin}}\boxed{130}\boxed{\text{ENTER}}$. The calculator should return the value .7660444431.

Example 6: Find the approximate value of each expression. Round your answer to 4 decimal places.

a. $\sin 130^\circ$

b. $\cos 15^\circ$

c. $\tan 50^\circ$

Solution:

a. $\sin 130^\circ \approx 0.7660$

b. $\cos 15^\circ \approx 0.9659$

c. $\tan 50^\circ \approx 1.1918$

You may have noticed that the calculator provides a "(" after the SIN. In the previous calculations, you can actually leave off the ")". However, in more complicated calculations, leaving off the closing ")" can create problems. It is a good idea to get in the habit of closing parentheses.

You can also use a calculator to find values of more complicated expressions.

Example 7: Use a calculator to find an approximate value of $\sin 25^\circ + \cos 25^\circ$. Round your answer to 4 decimal places.

Solution: $\sin 25^\circ + \cos 25^\circ \approx 1.3289$

*This is an example where you need to close the parentheses.

Points to Consider

- What is the difference between the measure of an angle, and its reference angle? In what cases are these measures the same value?
- Which angles have the same cosine value, or the same sine value? Which angles have opposite cosine and sine values?

Review Questions

1. State the reference angle for each angle.
   a. 190°
   b. -60°
   c. 1470°
   d. -135°
2. State the ordered pair for each angle.
   a. 300°
   b. -150°
   c. 405°
3. Find the value of each expression.
   a. $\sin 210^\circ$
   b. $\tan 270^\circ$
   c. $\csc 120^\circ$
4. Find the value of each expression.
   a. $\sin 510^\circ$
   b. $\cos 930^\circ$
   c. $\csc 405^\circ$
5. Find the value of each expression.
   a. $\cos -150^\circ$
   b. $\tan -45^\circ$
   c. $\sin -240^\circ$
6. Use a calculator to find each value. Round to 4 decimal places.
   a. $\sin 118^\circ$
   b. $\tan 55^\circ$
   c. $\cos 100^\circ$
7. Recall, in lesson 1.4, we introduced inverse trig functions. Use your calculator to find the measure of an angle whose sine value is 0.2.
8. In example 6c, we found that $\tan 50^\circ \approx 1.1918$. Use your knowledge of a special angle to explain why this value is reasonable. *HINT: You will need to change the tangent of this angle into a decimal.*

9. Use the table below or a calculator to explore sum and product relationships among trig functions. Consider the following functions:

$$f(x) = \sin(x+x) \text{ and } g(x) = \sin(x) + \sin(x)$$
$$h(x) = \sin(x) \,^*\, \sin(x) \text{ and } j(x) = \sin(x^2)$$

Do you observe any patterns in these functions? Are there any equalities among the functions? Can you make a general conjecture about $\sin(a) + \sin(b)$ and $\sin(a+b)$ for all values of a, b? What about $\sin(a)\sin(a)$ and $\sin(a^2)$?

TABLE 1.2:

$a°$	$b°$	$\sin a + \sin b$	$\sin(a+b)$
10	30	.6736	.6428
20	60	1.2080	.9848
55	78	1.7973	.7314
122	25	1.2707	.5446
200	75	.6239	-.9962

10. Use a calculator or your knowledge of special angles to fill in the values in the table, then use the values to make a conjecture about the relationship between $(\sin a)^2$ and $(\cos a)^2$. If you use a calculator, round all values to 4 decimal places.

TABLE 1.3:

a	$(\sin a)^2$	$(\cos a)^2$
0		
25		
45		
80		
90		
120		
250		

Review Answers

1.
   1. 10°
   2. 60°
   3. 30°
   4. 45°

   1. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
   2. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   3. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

   1. $-\frac{1}{2}$

2. 0
3. $\frac{2}{\sqrt{3}}$

1. $\frac{1}{2}$
2. $-\frac{\sqrt{3}}{2}$
3. $\sqrt{2}$

1. $-\frac{\sqrt{3}}{2}$
2. -1
3. $\frac{\sqrt{3}}{2}$

1. 0.8828
2. 1.4281
3. -0.1736

2. About 11.54 degrees or about 168.46 degrees.
3. This is reasonable because $\tan 45^\circ = 1$ and the $\tan 60^\circ = \sqrt{3} \approx 1.732$, and the $\tan 50^\circ$ should fall between these two values.
4. Conjecture: $\sin a + \sin b \neq \sin(a+b)$

TABLE 1.4:

a	$(\sin a)^2$	$(\cos a)^2$
0	0	1
25	.1786	.8216
45	$\frac{1}{2}$	$\frac{1}{2}$
80	.9698	.0302
90	1	0
120	.75	.25
235	.6710	.3290
310	.5898	.4132

Conjecture: $(\sin a)^2 + (\cos a)^2 = 1$.

1.8 Relating Trigonometric Functions

Learning Objectives

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.
- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.
- State the domain and range of each trig function.
- State the sign of a trig function, given the quadrant in which an angle lies.
- State the Pythagorean identities and use these identities to find values of trig functions.

Reciprocal identities

The first set of identities we will establish are the reciprocal identities. A **reciprocal** of a fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{or} \qquad \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \qquad \text{or} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Example 1: Find the value of each expression using a reciprocal identity.

a. $\cos\theta = .3, \sec\theta = ?$

b. $\cot\theta = \frac{4}{3}, \tan\theta = ?$

Solution:

a. $\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = \frac{10}{3}$.

b. $\tan\theta = \frac{3}{4}$

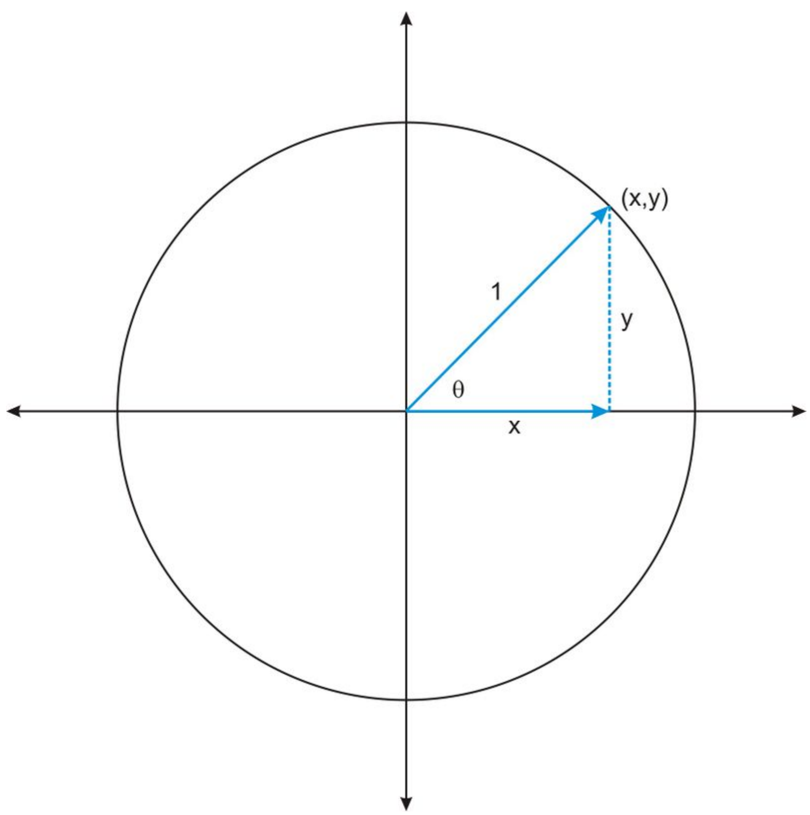
These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

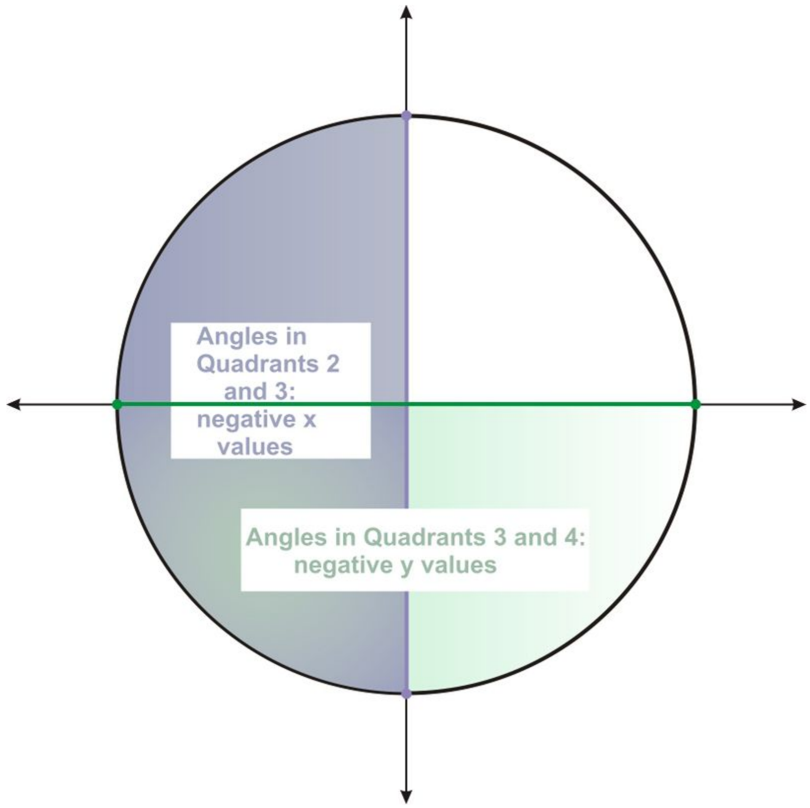
Domain, Range, and Signs of Trig Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, R. We can determine the range of the functions if we think about the fact that the sine of an angle is the $y-$coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $x-$coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.



In this right triangle, x and y are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative $y-$coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative $x-$coordinate.



In either case, the minimum value is -1. For example, $\cos 180^\circ = -1$ and $\sin 270^\circ = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

TABLE 1.5:

	Domain	Range
Sine	$\theta = R$	$-1 \leq y \leq 1$
Cosine	$\theta = R$	$-1 \leq y \leq 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

TABLE 1.6:

	Domain	Range
Cosecant	$\theta \varepsilon R, \theta \neq 0, 180, 360\ldots$	$\csc\theta \leq -1$ or $\csc\theta \geq 1$
Secant	$\theta \varepsilon R, \theta \neq 90, 270, 450\ldots$	$\sec\theta \leq -1$ or $\sec\theta \geq 1$

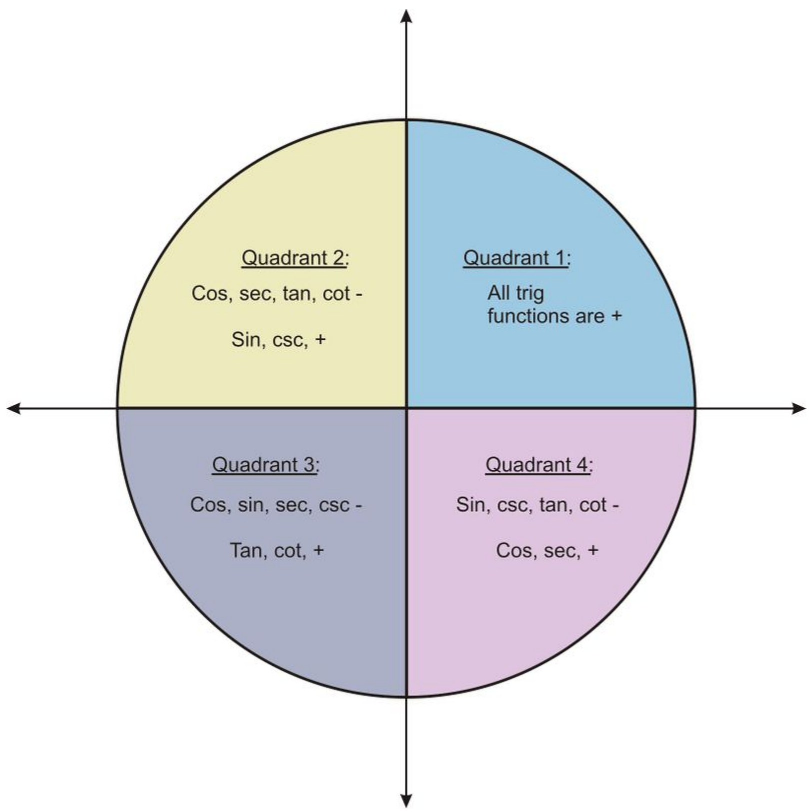
Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan\theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an $x-$coordinate of $0 : 90^\circ, 270^\circ$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a $y-$coordinate of $0 : 0°$, $180°$, $360°$, etc.

TABLE 1.7:

Function	Domain	Range
Tangent	$\theta \varepsilon R, \theta \neq 90, 270, 450\ldots$	All reals
Cotangent	$\theta \varepsilon R, \theta \neq 0, 180, 360\ldots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.



An easy way to remember this is "All Students Take Calculus." Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which trig functions are positive and where.

Example 2: State the sign of each expression.

a. $\cos 100°$

b. $\csc 220°$

c. $\tan 370°$

Solution:

a. The angle $100°$ is in the second quadrant. Therefore the $x-$coordinate is negative and so $\cos 100°$ is negative.

b. The angle 220° is in the third quadrant. Therefore the $y-$coordinate is negative. So the sine, and the cosecant are negative.

c. The angle 370° is in the first quadrant. Therefore the tangent value is positive.

So far we have considered relationships between pairs of functions: the six trig functions can be grouped in pairs as reciprocals. Now we will consider relationships among three trig functions.

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities above. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example 3: If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

Solution: $\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example 4: Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solution:

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

This is also an identity that you can use to find the value of the cotangent function, given values of sine and cosine. Both of the quotient identities will also be useful in chapter 3, in which you will prove other identities.

Cofunction Identities and Reflection

These identities relate to the problems you did in section 1.3. Recall, #3 and #4 from the review questions, where $\sin X = \cos Z$ and $\cos X = \sin Z$, where X and Z were complementary angles. These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$
$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Example 5: Find the value of each trig function.

a. $\cos 120^\circ$

b. $\cos(-120^\circ)$

c. $\sin 135^\circ$

d. $\sin(-135^\circ)$

Solution: Because these angles have reference angles of 60° and 45°, the values are:

a. $\cos 120^\circ = -\frac{1}{2}$

b. $\cos(-120^\circ) = \cos 240^\circ = -\frac{1}{2}$

c. $\sin 135^\circ = \frac{\sqrt{2}}{2}$

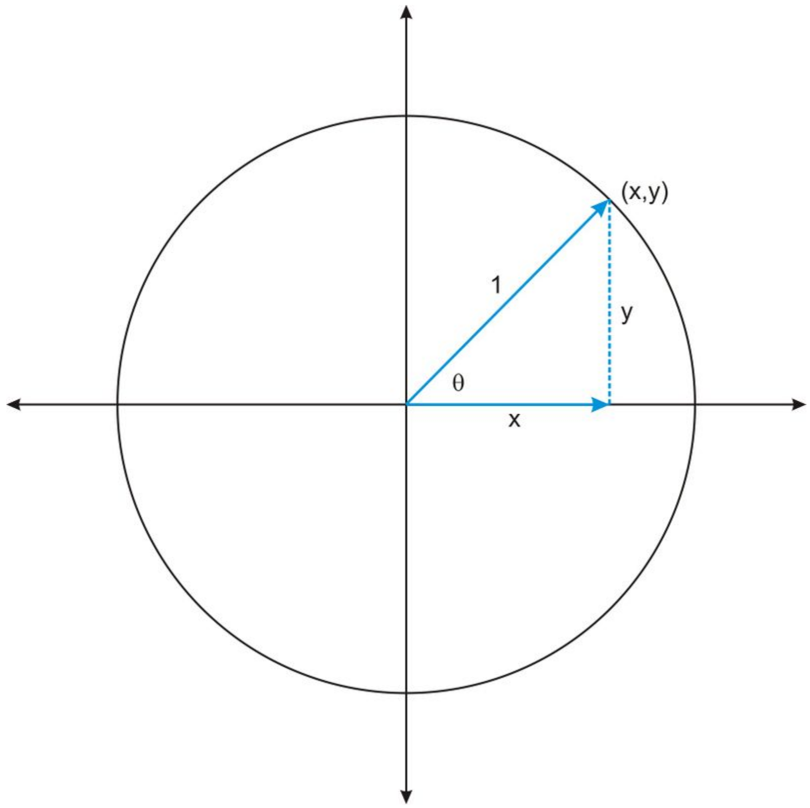
d. $\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2}$

These values show us that sine and cosine also reflect over the x axis. This allows us to generate three more identities.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Pythagorean Identities

The final set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In previous lessons we used the Pythagorean Theorem to find the sides of right triangles. Consider once again the way that we defined the trig functions in 1.3. Let's look at the unit circle:



The legs of the right triangle are x, and y. The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ *means* $(\cos\theta)^2$ *and* $\sin^2\theta$ *means* $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example 6: If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\left(\frac{1}{4}\right)^2 + \sin^2\theta &= 1\\
\frac{1}{16} + \sin^2\theta &= 1\\
\sin^2\theta &= 1 - \frac{1}{16}\\
\sin^2\theta &= \frac{15}{16}\\
\sin\theta &= \pm\sqrt{\frac{15}{16}}\\
\sin\theta &= \pm\frac{\sqrt{15}}{4}
\end{aligned}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example 7: Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that $\cot^2\theta + 1 = \csc^2\theta$

Solution:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 && \text{Divide both sides by} \sin^2\theta.\\
\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} && \frac{\sin^2\theta}{\sin^2\theta} = 1\\
\frac{\cos^2\theta}{\sin^2\theta} + 1 &= \frac{1}{\sin^2\theta}\\
\frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} + 1 &= \frac{1}{\sin\theta} \times \frac{1}{\sin\theta} && \text{Write the squared functions in terms of their factors.}\\
\cot\theta \times \cot\theta + 1 &= \csc\theta \times \csc\theta && \text{Use the quotient and reciprocal identities.}\\
\cot^2\theta + 1 &= \csc^2\theta && \text{Write the functions as squared functions.}
\end{aligned}$$

Points to Consider

1. How do you know if an equation is an identity? *HINT: you could consider using a the calculator and graphing a related function, or you could try to prove it mathematically.*
2. How can you verify the domain or range of a function?

Review Questions

1. Use reciprocal identities to give the value of each expression.
   a. $\sec\theta = 4, \cos\theta = ?$
   b. $\sin\theta = \frac{1}{3}, \csc\theta = ?$
2. In the lesson, the range of the cosecant function was given as: $\csc\theta \le -1$ or $\csc\theta \ge 1$.
   a. Use a calculator to fill in the table below. Round values to 4 decimal places.
   b. Use the values in the table to explain in your own words what happens to the values of the cosecant function as the measure of the angle approaches 0 degrees.
   c. Explain what this tells you about the range of the cosecant function.
   d. Discuss how you might further explore values of the sine and cosecant to better understand the range of the cosecant function.

TABLE 1.8:

Angle	Sin	Csc
10		
5		
1		
0.5		
0.1		
0		
-.1		
-.5		
-1		
-5		
-10		

3. In the lesson the domain of the secant function were given: Domain: $\theta\varepsilon^\circ, \theta \neq 90, 270, 450\ldots$ Explain why certain values are excluded from the domain.
4. State the quadrant in which each angle lies, and state the sign of each expression
   a. $\sin 80^\circ$
   b. $\cos 200^\circ$
   c. $\cot 325^\circ$
   d. $\tan 110^\circ$
5. If $\cos\theta = \frac{6}{10}$ and $\sin\theta = \frac{8}{10}$, what is the value of $\tan\theta$?
6. Use quotient identities to explain why the tangent and cotangent function have positive values for angles in the third quadrant.
7. If $\sin\theta = 0.4$, what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.
8. If $\cot\theta = 2$, what is the value of $\csc\theta$? Assume that θ is an angle in the first quadrant.
9. Show that $1 + \tan^2\theta = \sec^2\theta$.
10. Explain why it is necessary to state the quadrant in which the angle lies for problems such as #7.

Review Answers

1. 1. $\frac{1}{4}$

2. $\frac{3}{1} = 3$

2. (a)

TABLE 1.9:

Angle	Sin	Csc
10	.1737	5.759
5	.0872	11.4737
1	.0175	57.2987
0.5	.0087	114.5930
0.1	.0018	572.9581
0	0	undefined
-.1	-.0018	-572.9581
-.5	-.0087	-114.5930
-1	-.0175	-57.2987
-5	-.0872	-11.4737
-10	-.1737	-5.759

(b) As the angle gets smaller and smaller, the cosecant values get larger and larger.

(c) The range of the cosecant function does not have a maximum, like the sine function. The values get larger and larger.

(d) Answers will vary. For example, if we looked at values near 90 degrees, we would see the cosecant values get smaller and smaller, approaching 1.

3. The values 90, 270, 450, etc, are excluded because they make the function undefined.
   1. Quadrant 1; positive
   2. Quadrant 3; negative
   3. Quadrant 4; negative
   4. Quadrant 2; negative
4. $\frac{8}{6} = \frac{4}{3}$
5. The ratio of sine and cosine will be positive in the third quadrant because sine and cosine are both negative in the third quadrant.
6. $\cos\theta \approx .92$
7. $\csc\theta = \sqrt{5}$
8.

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
1 + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
1 + \tan^2\theta &= \sec^2\theta
\end{aligned}$$

9. Using the Pythagorean identities results in a quadratic equation and will have two solutions. Stating that the angle lies in a particular quadrant tells you which solution is the actual value of the expression. In #7, the angle is in the first quadrant, so both sine and cosine must be positive.

Chapter Summary

In this chapter students learned about right triangles and special right triangles. Through the special right triangles and the Pythagorean Theorem, the study of trigonometry was discovered. Sine, cosine, tangent, secant, cosecant, and cotangent are all functions of angles and the result is the ratio of the sides of a right triangle. We learned that only our special right triangles generate sine, cosine, tangent values that can be found without the use of a scientific calculator. When incorporating the trig ratios and the Pythagorean Theorem, we discovered the first of many trig identities. This concept will be explored further in Chapter 3.

Vocabulary

Adjacent
A side adjacent to an angle is the side next to the angle. In a right triangle, it is the leg that is next to the angle.

Angle of depression
The angle between the horizontal line of sight, and the line of sight down to a given point.

Angle of elevation
The angle between the horizontal line of sight, and the line of sight up to a given point.

Bearings
The direction from one object to another, usually measured as an angle.

Clinometer
A device used to measure angles of elevation or depression.

Coterminal angles
Two angles in standard position are coterminal if they share the same terminal side.

Distance Formula
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Hypotenuse
The hypotenuse is the longest side in a right triangle, opposite the right angle.

Identity
An identity is an equation that is always true, as long as the variables and expressions involved are defined.

Included Angle
The angle inbetween two sides of a polygon.

Leg
The legs of a right triangle are the two shorter sides.

Nautical Mile
A nautical mile is a unit of length that corresponds approximately to one minute of latitude along any meridian. A nautical mile is equal to 1.852 meters.

Pythagorean Theorem

$a^2 + b^2 = c^2$

Pythagorean Triple

A set whole numbers for which the Pythagorean Theorem holds true.

Quadrantal angle

A quadrantal angle is an angle in standard position whose terminal side lies on an axis.

Radius

The radius of a circle is the distance from the center of the circle to the edge. The radius defines the circle.

Reference angle

The reference angle of an angle in standard position is the measure of the angle between the terminal side and the closest portion of the $x-$axis.

Standard position

An angle in standard position has its initial side on the positive $x-$axis, its vertex at the origin, and its terminal side anywhere in the plane. A positive angle means a counterclockwise rotation. A negative angle means a clockwise rotation.

Theodolite

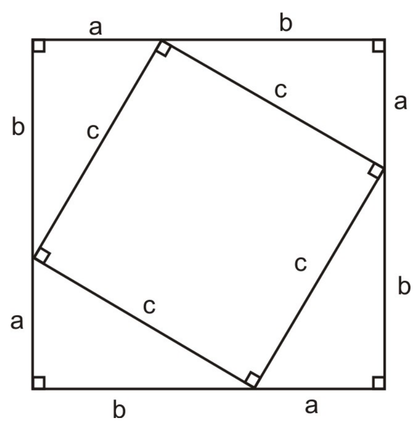
A device used to measure angles of elevation or depression.

Unit Circle

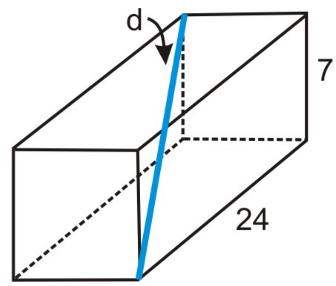
The unit circle is the circle with radius 1 and center (0, 0). The equation of the unit circle is $x^2 + y^2 = 1$

Review Questions

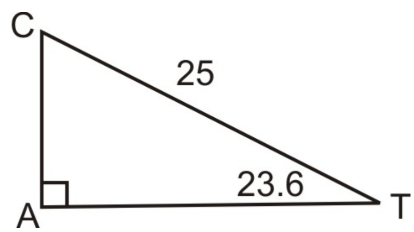
1. One way to prove the Pythagorean Theorem is through the picture below. Determine the area of the square two different ways and set each equal to each other.



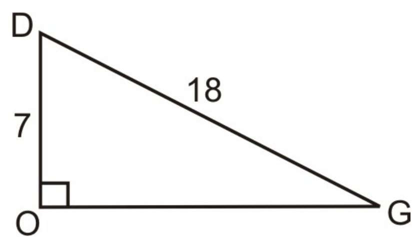
2. A flute is resting diagonally, d, in the rectangular box (prism) below. Find the length of the flute.



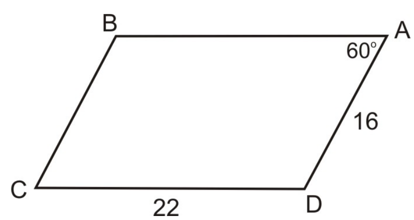
3. Solve the right triangle.



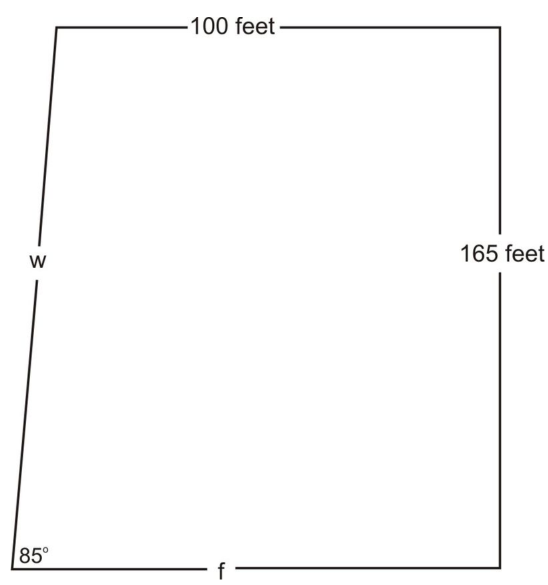
4. Solve the right triangle.



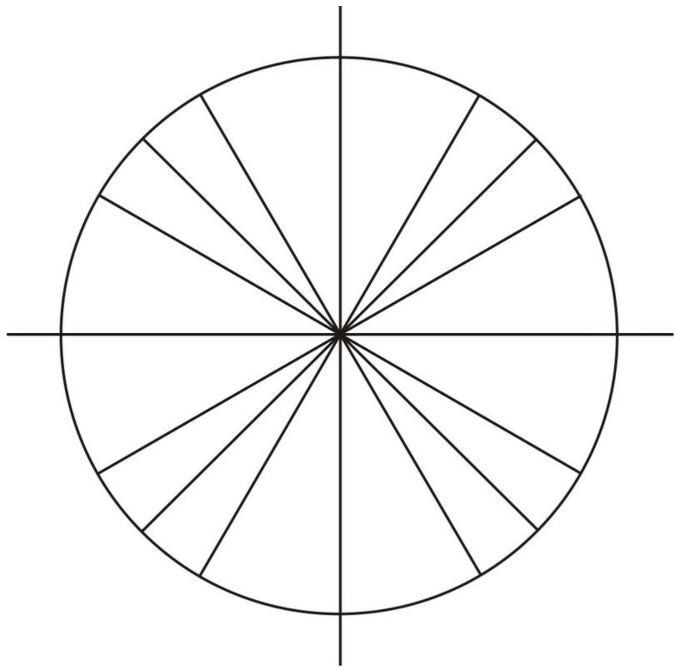
5. Find the *exact* value of the area of the parallelogram below.



6. The modern building shown below is built with an outer wall (shown on the left) that is not at a 90-degree angle with the floor. The wall on the right is perpendicular to both the floor and ceiling. Find the length of *w*.



7. Given that $\cos(90^\circ - x) = \frac{2}{7}$, find the $\sin x$.
8. If $\cos(-x) = \frac{3}{4}$ and $\tan x = \frac{\sqrt{7}}{3}$, find $\sin(-x)$.
9. If $\sin y = \frac{1}{3}$, what is $\cos y$?
10. $\sin\theta = \frac{1}{3}$ find the value(s) of $\cos\theta$.
11. $\cos\theta = -\frac{2}{5}$, and θ is a second quadrant angle. Find the exact values of remaining trigonometric functions.
12. (3, -4) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.
13. Determine reference angle and two coterminal angles for 165°. Plot the angle in standard position.
14. It is very helpful to have the unit circle with all the special values on one circle. Fill out the unit circle below with all of the endpoints for each special value and quadrantal value.



Review Answers

1. Area 1:

$$
\begin{gathered}
(a+b)^2 \\
(a+b)(a+b) \\
a^2+2ab+b^2
\end{gathered}
$$

Area 2: Add up 4 triangles and inner square.

$$
\begin{aligned}
4 \cdot \frac{1}{2}ab + c^2 \\
2ab + c^2
\end{aligned}
$$

Set the two equal to each other:

$$
\begin{gathered}
a^2+2ab+b^2 = 2ab+c^2 \\
a^2+b^2 = c^2
\end{gathered}
$$

2. First, find the diagonal of the base. This is a Pythagorean Triple, so the base diagonal is 25 (you could have also done the Pythagorean Theorem if you didn't see this). Now, do the Pythagorean Theorem with the height and the diagonal to get the three-dimensional diagonal.

$$\begin{aligned} 7^2 + 25^2 &= d^2 \\ 49 + 225 &= d^2 \\ 274 &= d^2 \\ \sqrt{274} &= d \approx 16.55 \end{aligned}$$

3.

$$\angle C = 90^\circ - 23.6^\circ = 66.4^\circ$$

$$\begin{aligned} \sin 23.6 &= \frac{CA}{25} & \cos 23.6 &= \frac{AT}{25} \\ 25 \cdot \sin 23.6 &= CA & 25 \cdot \cos 23.6 &= AT \\ 10.01 &\approx CA & 22.9 &\approx AT \end{aligned}$$

4. First do the Pythagorean Theorem to get the third side.

$$\begin{aligned} 7^2 + x^2 &= 18^2 \\ 49 + x^2 &= 324 \\ x^2 &= 275 \\ x &= \sqrt{275} = 5\sqrt{11} \end{aligned}$$

Second, use one of the inverse functions to find the two missing angles.

$$\begin{aligned} \sin G &= \frac{7}{18} \\ \sin^{-1}\left(\frac{7}{18}\right) &= G \\ G &\approx 22.89^\circ \end{aligned}$$

We can subtract $\angle G$ from 90 to get 67.11°.

5.

$$\begin{aligned} A &= ab \sin C \\ &= 16 \cdot 22 \cdot \sin 60^\circ \\ &= 352 \cdot \frac{\sqrt{3}}{2} \\ &= 176\sqrt{3} \end{aligned}$$

6. Make a right triangle with 165 as the opposite leg and w is the hypotenuse.

$$\begin{aligned} \sin 85^\circ &= \frac{165}{w} \\ w \sin 85^\circ &= 165 \\ w &= \frac{165}{\sin 85^\circ} \\ w &\approx 165.63 \end{aligned}$$

7.

$$\begin{aligned} \cos(90^\circ - x) &= \sin x \\ \sin x &= \frac{2}{7} \end{aligned}$$

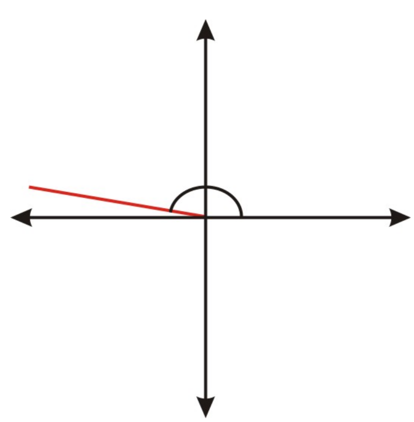
8. If $\cos(-x) = \frac{3}{4}$, then $\cos x = \frac{3}{4}$. With $\tan x = \frac{\sqrt{7}}{3}$, we can conclude that $\sin x = \frac{\sqrt{7}}{4}$ and $\sin(-x) = -\frac{\sqrt{7}}{4}$.
9. If $\sin y = \frac{1}{3}$, then we know the opposite side and the hypotenuse. Using the Pythagorean Theorem, we get that the adjacent side is $2\sqrt{2}\left(1^2 + b^2 = 3^2 \rightarrow b = \sqrt{9-1} \rightarrow b = \sqrt{8} = 2\sqrt{2}\right)$. Thus, $\cos y = \pm\frac{2\sqrt{2}}{3}$ because we don't know if the angle is in the second or third quadrant.
10. $\sin\theta = \frac{1}{3}$, sine is positive in Quadrants I and II. So, there can be two possible answers for the $\cos\theta$. Find the third side, using the Pythagorean Theorem:

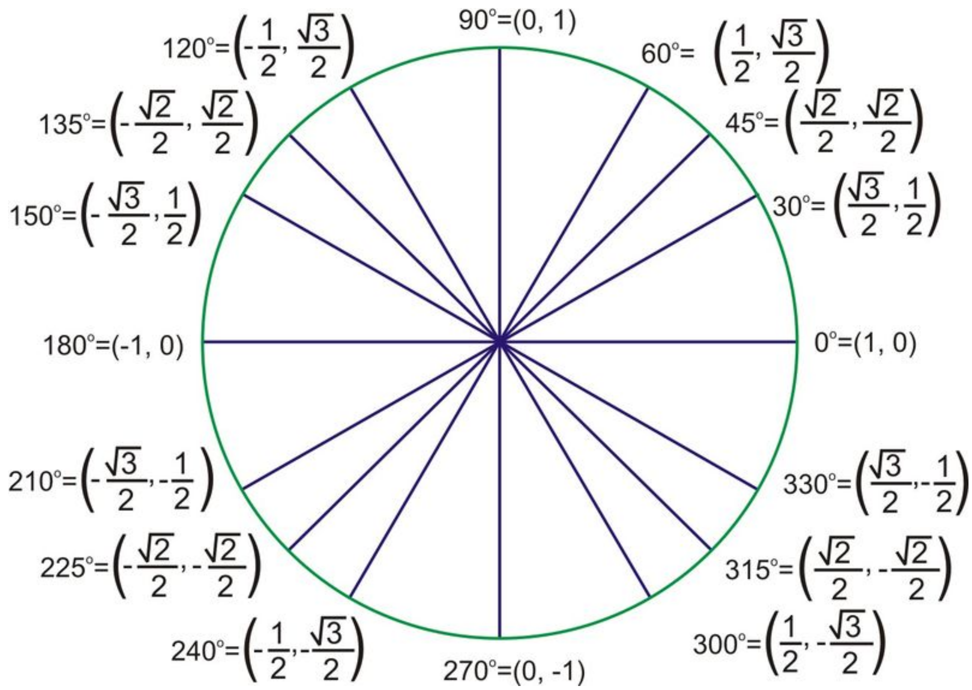
$$\begin{aligned} 1^2 + b^2 &= 3^2 \\ 1 + b^2 &= 9 \\ b^2 &= 8 \\ b &= \sqrt{8} = 2\sqrt{2} \end{aligned}$$

In Quadrant I, $\cos\theta = \frac{2\sqrt{2}}{3}$ In Quadrant II, $\cos\theta = -\frac{2\sqrt{2}}{3}$
11. $\cos\theta = -\frac{2}{5}$ and is in Quadrant II, so from the Pythagorean Theorem :

$$\begin{aligned} a^2 + (-2)^2 &= 5^2 \\ a^2 + 4 &= 25 \\ a^2 &= 21 \\ a &= \sqrt{21} \end{aligned}$$

So, $\sin\theta = \frac{\sqrt{21}}{5}$ and $\tan\theta = -\frac{\sqrt{21}}{2}$
12. If the terminal side of θ is on (3, -4) means θ is in Quadrant IV, so cosine is the only positive function. Because the two legs are lengths 3 and 4, we know that the hypotenuse is 5. 3, 4, 5 is a Pythagorean Triple (you can do the Pythagorean Theorem to verify). Therefore, $\sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}, \tan\theta = -\frac{4}{3}$
13. Reference angle $= 15^\circ$. Possible coterminal angles $= -195^\circ, 525^\circ$



14. 

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9699.