Developmental Math—An Open Program
Unit – Geometry
First Edition

Lesson 1 – Basic Geometric Concepts and Figures

TOPICS

1.1.1 Figures in 1 and 2 Dimensions
1 Identify and define points, lines, line segments, rays, and planes.
2 Classify angles as acute, right, obtuse, or straight.

1.1.2 Properties of Angles
1 Identify parallel and perpendicular lines.
2 Find measures of angles.
3 Identify complementary and supplementary angles.

1.1.3 Triangles
1 Identify equilateral, isosceles, scalene, acute, right, and obtuse triangles.
2 Identify whether triangles are similar, congruent, or neither.
3 Identify corresponding sides of congruent and similar triangles.
4 Find the missing measurements in a pair of similar triangles.
5 Solve application problems involving similar triangles.

1.1.4 The Pythagorean Theorem
1 Use the Pythagorean Theorem to find the unknown side of a right triangle.
2 Solve application problems involving the Pythagorean Theorem.

Lesson 2 – Perimeter, Circumference, and Area

TOPICS

1.2.1 Quadrilaterals
1 Identify properties, including angle measurements, of quadrilaterals.

Lesson 2 – Perimeter, Circumference, and Area

TOPICS (continued)

1.2.2 Perimeter and Area

1 Find the perimeter of a polygon.
2 Find the area of a polygon.
3 Find the area and perimeter of non-standard polygons.

1.2.3 Circles

1 Identify properties of circles.
2 Find the circumference of a circle.
3 Find the area of a circle.
4 Find the area and perimeter of composite geometric figures.

Lesson 3 – Volume of Geometric Solids

TOPICS

1.3.1 Solids

1 Identify geometric solids.
2 Find the volume of geometric solids.
3 Find the volume of a composite geometric solid.

1.1.1 Figures in 1 and 2 Dimensions

Learning Objective(s)
1 Identify and define points, lines, line segments, rays and planes.
2 Classify angles as acute, right, obtuse, or straight.

Introduction

You use geometric terms in everyday language, often without thinking about it. For example, any time you say “walk along this line” or “watch out, this road quickly angles to the left” you are using geometric terms to make sense of the environment around you. You use these terms flexibly, and people generally know what you are talking about.

In the world of mathematics, each of these geometric terms has a specific definition. It is important to know these definitions—as well as how different figures are constructed—to become familiar with the language of geometry. Let’s start with a basic geometric figure: the plane.

Figures on a Plane

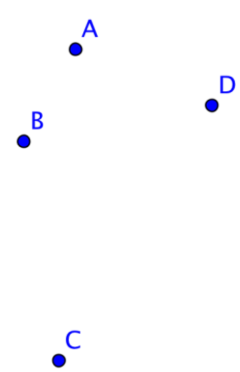
Objective **1**

A **plane** is a flat surface that continues forever (or, in mathematical terms, infinitely) in every direction. It has two dimensions: length and width.

You can visualize a plane by placing a piece of paper on a table. Now imagine that the piece of paper stays perfectly flat and extends as far as you can see in two directions, left-to-right and front-to-back. This gigantic piece of paper gives you a sense of what a geometric plane is like: it continues infinitely in two directions. (Unlike the piece of paper example, though, a geometric plane has no height.)

A plane can contain a number of geometric figures. The most basic geometric idea is a **point**, which has no dimensions. A point is simply a location on the plane. It is represented by a dot. Three points that don’t lie in a straight line will determine a plane.

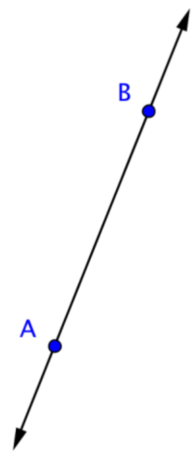
The image below shows four points, labeled *A*, *B*, *C*, and *D*.



Two points on a plane determine a line. A **line** is a one-dimensional figure that is made up of an infinite number of individual points placed side by side. In geometry all lines are assumed to be straight; if they bend they are called a curve. A line continues infinitely in two directions.

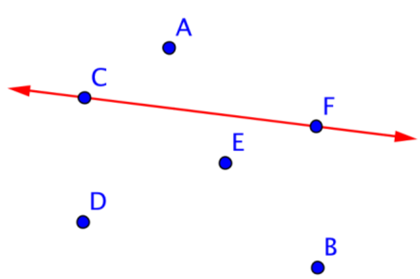
Below is line *AB* or, in geometric notation, $\overleftrightarrow{AB}$. The arrows indicate that the line keeps going forever in the two directions. This line could also be called line *BA*. While the order of the points does not matter for a line, it is customary to name the two points in alphabetical order.

The image below shows the points *A* and *B* and the line $\overleftrightarrow{AB}$.



Example

Problem **Name the line shown in red.**



The red line goes through the points *C* and *F*, so the line is $\overleftrightarrow{CF}$.

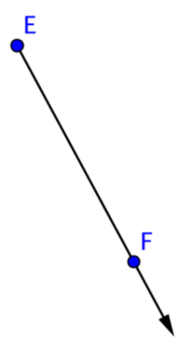
Answer $\overleftrightarrow{CF}$

There are two more figures to consider. The section between any two points on a line is called a **line segment**. A line segment can be very long, very short, or somewhere in between. The difference between a line and a line segment is that the line segment has two endpoints and a line goes on forever. A line segment is denoted by its two endpoints, as in $\overline{CD}$.

A **ray** has one endpoint and goes on forever in one direction. Mathematicians name a ray with notation like $\overrightarrow{EF}$, where point *E* is the endpoint and *F* is a point on the ray. When naming a ray, we always say the endpoint first. Note that $\overrightarrow{FE}$ would have the endpoint at *F*, and continue through *E*, which is a different ray than $\overrightarrow{EF}$, which would have an endpoint at E, and continue through F.

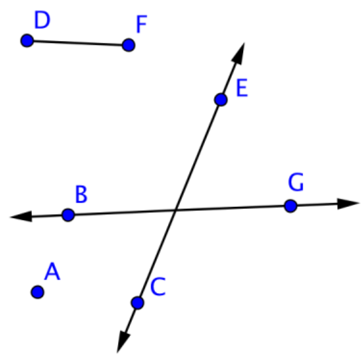
The term “ray” may be familiar because it is a common word in English. “Ray” is often used when talking about light. While a ray of light resembles the geometric term “ray,” it does not go on forever, and it has some width. A geometric ray has no width; only length.

Below is an image of ray *EF* or $\overrightarrow{EF}$. Notice that the end point is *E*.



Example

Problem **Identify each line and line segment in the picture below.**



Two points define a line, and a line is denoted with arrows. There are two lines in this picture: $\overleftrightarrow{CE}$ and $\overleftrightarrow{BG}$.

A line segment is a section between two points. $\overline{DF}$ is a line segment. But there are also two more line segments on the lines themselves: $\overline{CE}$ and $\overline{BG}$.

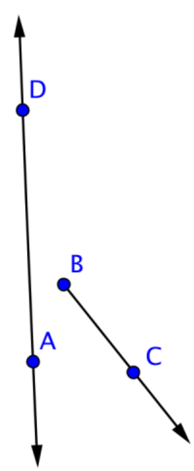
Answer

Lines: $\overleftrightarrow{CE}, \overleftrightarrow{BG}$

Line segments: $\overline{DF}, \overline{CE}, \overline{BG}$

Example

Problem **Identify each point and ray in the picture below.**



There are four points: *A*, *B*, *C*, and *D*. There are also three rays, though only one may be obvious.

Ray $\overrightarrow{BC}$ begins at point *B* and goes through *C*. Two more rays exist on line $\overleftrightarrow{AD}$: they are $\overrightarrow{DA}$ and $\overrightarrow{AD}$.

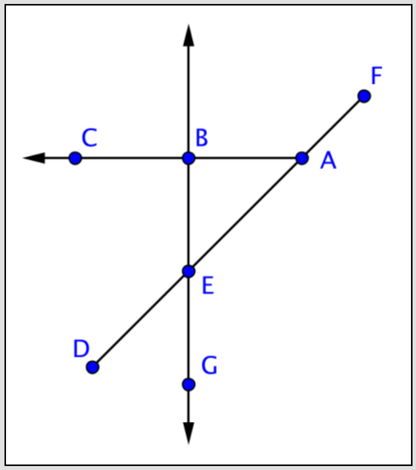
Answer

Points: *A*, *B*, *C*, *D*

Rays: $\overrightarrow{BC}, \overrightarrow{AD}, \overrightarrow{DA}$

Self Check A

Which of the following is *not* represented in the image below?



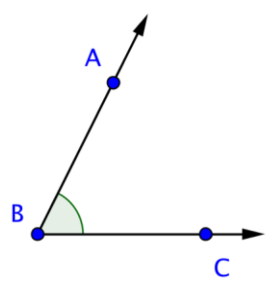
A) $\overleftrightarrow{BG}$
B) $\overrightarrow{BA}$
C) $\overline{DF}$
D) $\overrightarrow{AC}$

Angles

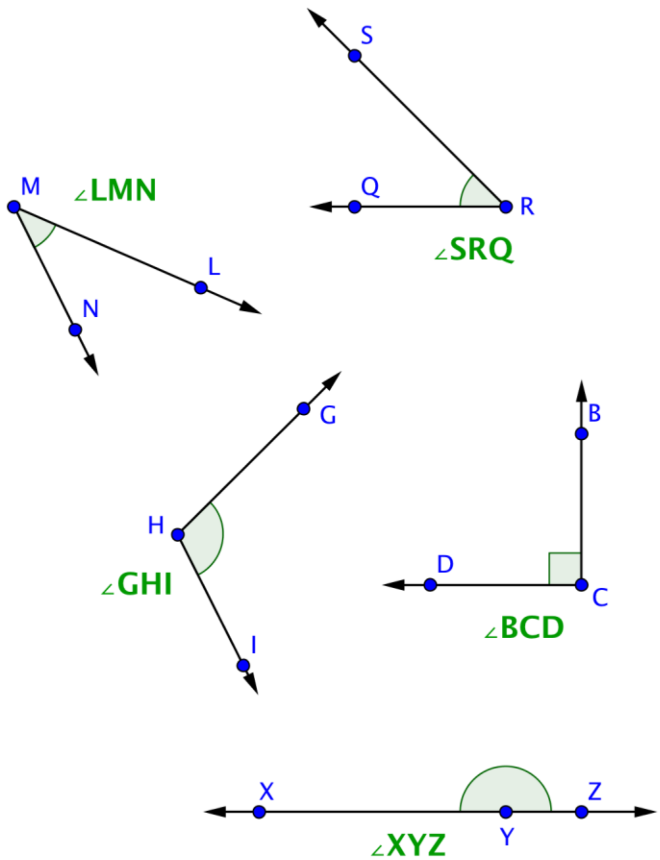
Objective **2**

Lines, line segments, points, and rays are the building blocks of other figures. For example, two rays with a common endpoint make up an **angle**. The common endpoint of the angle is called the **vertex**.

The angle *ABC* is shown below. This angle can also be called $\angle ABC$, $\angle CBA$ or simply $\angle B$. When you are naming angles, be careful to include the vertex (here, point *B*) as the middle letter.



The image below shows a few angles on a plane. Notice that the label of each angle is written "point-vertex-point," and the geometric notation is in the form $\angle ABC$.



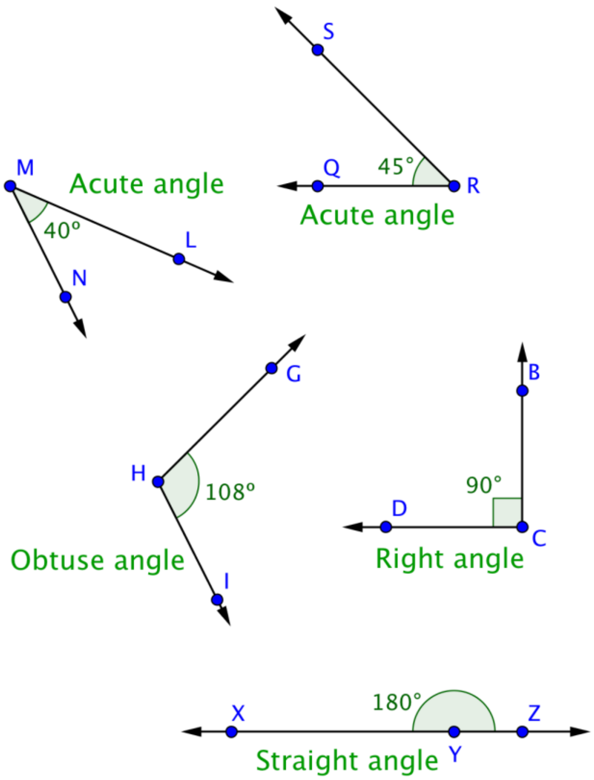
Sometimes angles are very narrow; sometimes they are very wide. When people talk about the "size" of an angle, they are referring to the arc between the two rays. The length of the rays has nothing to do with the size of the angle itself. Drawings of angles will often include an arc (as shown above) to help the reader identify the correct 'side' of the angle.

Think about an analog clock face. The minute and hour hands are both fixed at a point in the middle of the clock. As time passes, the hands rotate around the fixed point, making larger and smaller angles as they go. The length of the hands does not impact the angle that is made by the hands.

An angle is measured in degrees, represented by the symbol º. A circle is defined as having 360º. (In skateboarding and basketball, "doing a 360" refers to jumping and doing one complete body rotation.

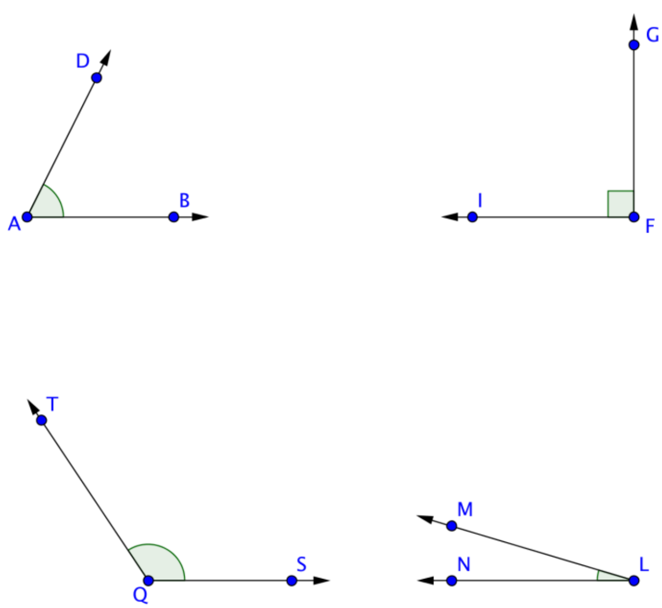
A **right angle** is any degree that measures exactly 90º. This represents exactly one-quarter of the way around a circle. Rectangles contain exactly four right angles. A corner mark is often used to denote a right angle, as shown in right angle DCB below.

Angles that are between 0º and 90º (smaller than right angles) are called **acute angles**. Angles that are between 90º and 180º (larger than right angles and less than 180º) are called **obtuse angles**. And an angle that measures exactly 180º is called a **straight angle** because it forms a straight line!



Example

Problem **Label each angle below as acute, right, or obtuse.**



You can start by identifying any right angles.

$\angle GFI$ is a right angle, as indicated by the corner mark at vertex *F*.

Acute angles will be smaller than $\angle GFI$ (or less than 90º). This means that $\angle DAB$ and $\angle MLN$ are acute.

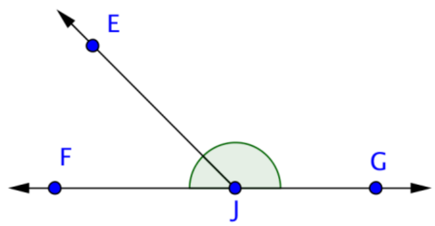
$\angle TQS$ is larger than $\angle GFI$, so it is an obtuse angle.

Answer

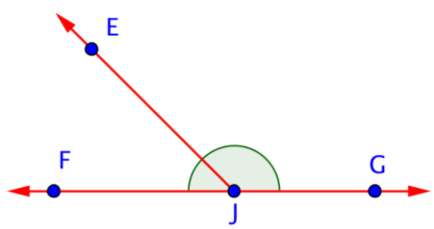
$\angle DAB$ and $\angle MLN$ are acute angles.
$\angle GFI$ is a right angle.
$\angle TQS$ is an obtuse angle.

Example

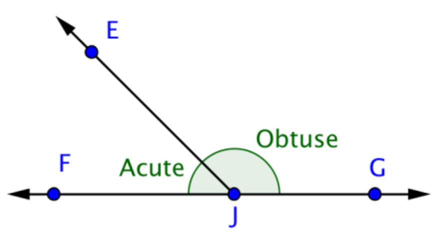
Problem **Identify each point, ray, and angle in the picture below.**



Begin by identifying each point in the figure. There are 4: *E*, *F*, *G*, and *J*.



Now find rays. A ray begins at one point, and then continues through another point towards infinity (indicated by an arrow). Three rays start at point *J*: $\overrightarrow{JE}$, $\overrightarrow{JF}$, and $\overrightarrow{JG}$. But also notice that a ray could start at point *F* and go through *J* and *G*, and another could start at point *G* and go through *J* and *F*. These rays can be represented by $\overrightarrow{GF}$ and $\overrightarrow{FG}$.



Finally, look for angles. $\angle EJG$ is obtuse, $\angle EJF$ is acute, and $\angle FJG$ is straight. (Don't forget those straight angles!)

Answer

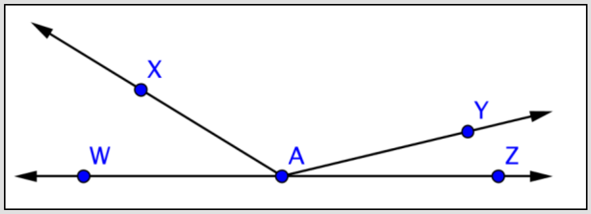
Points: *E*, *F*, *G*, *J*

Rays: $\overrightarrow{JE}, \overrightarrow{JG}, \overrightarrow{JF}, \overrightarrow{GF}, \overrightarrow{FG}$

Angles: $\angle EJG, \angle EJF, \angle FJG$

Self Check B

Identify the acute angles in the image below.



A) $\angle WAX$, $\angle XAY$, and $\angle YAZ$
B) $\angle WAY$ and $\angle YAZ$
C) $\angle WAX$ and $\angle YAZ$
D) $\angle WAZ$ and $\angle XAY$

Measuring Angles with a Protractor

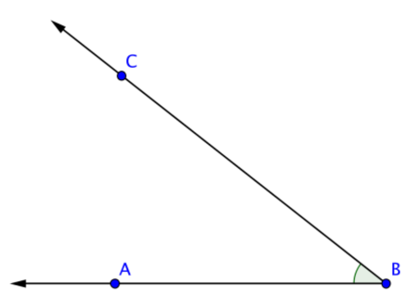
Learning how to measure angles can help you become more comfortable identifying the difference between angle measurements. For instance, how is a 135º angle different from a 45º angle?

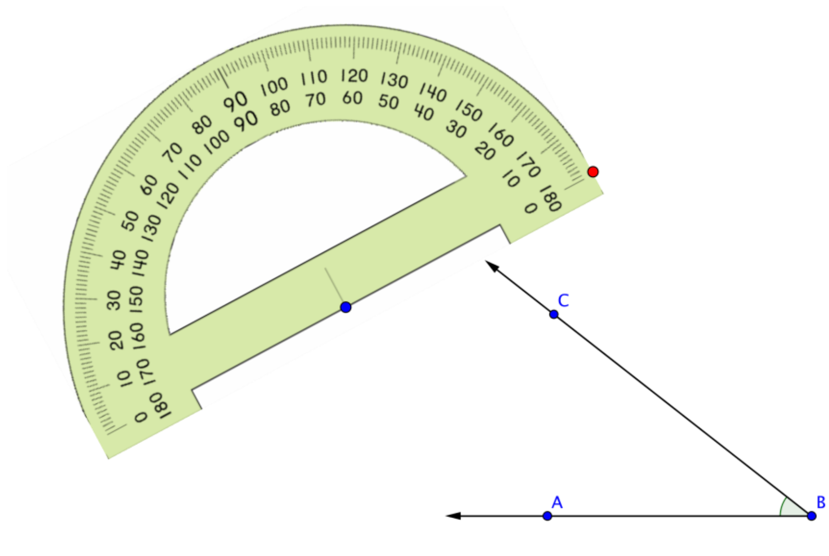
Measuring angles requires a **protractor**, which is a semi-circular tool containing 180 individual hash marks. Each hash mark represents 1º. (Think of it like this: a circle is 360º, so a semi-circle is 180º.) To use the protractor, do the following three steps:

1. line up the vertex of the angle with the dot in the middle of the flat side (bottom) of the protractor,
2. align one side of the angle with the line on the protractor that is at the zero degree mark, and
3. look at the curved section of the protractor to read the measurement.

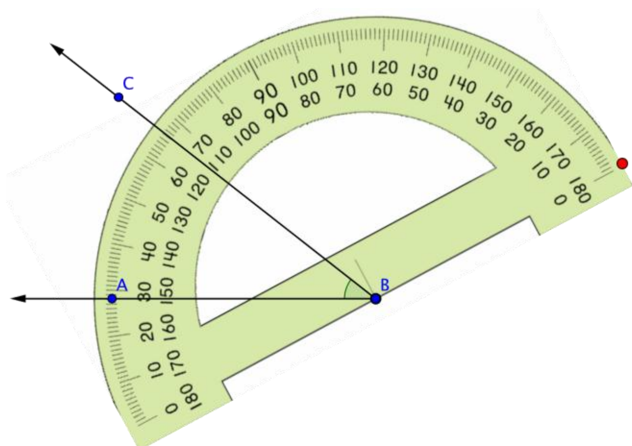
The example below shows you how to use a protractor to measure the size of an angle.

Example	
Problem	**Use a protractor to measure the angle shown below.**

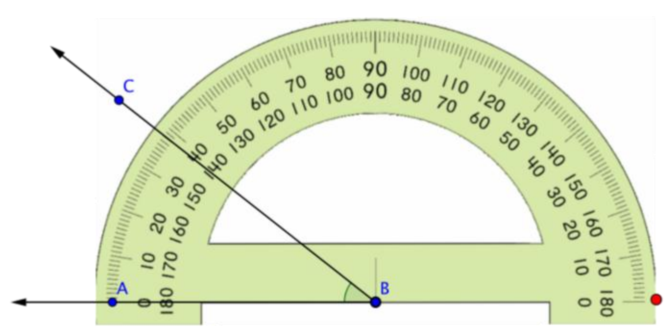




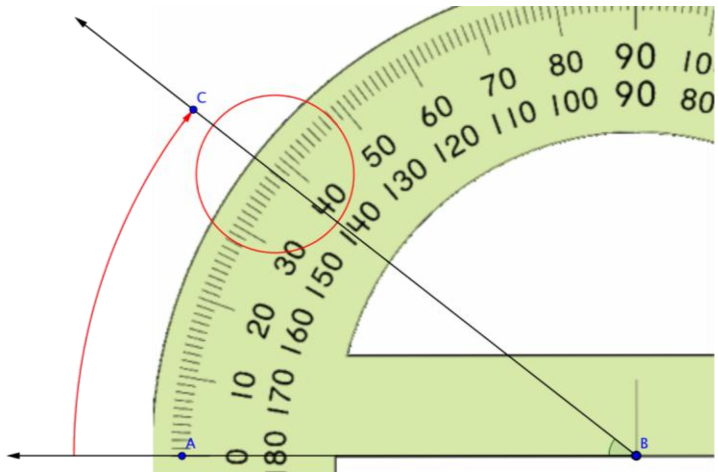
Use a protractor to measure the angle.



Align the blue dot on the protractor with the vertex of the angle you want to measure.



Rotate the protractor around the vertex of the angle until the side of the angle is aligned with the 0 degree mark of the protractor.

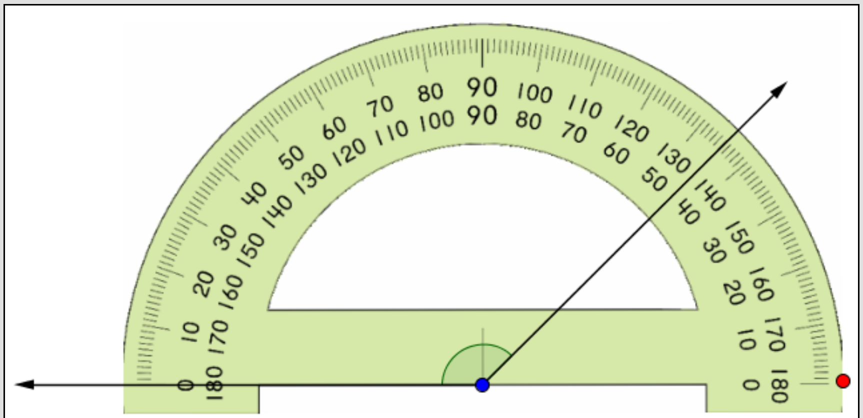


Read the measurement, in degrees, of the angle. Begin with the side of the angle that is aligned with the 0º mark of the protractor and count up from 0º. This angle measures 38º.

Answer The angle measures 38º.

Self Check C

What is the measurement of the angle shown below?



A) 45º
B) 135º
C) 145º
D) 180º

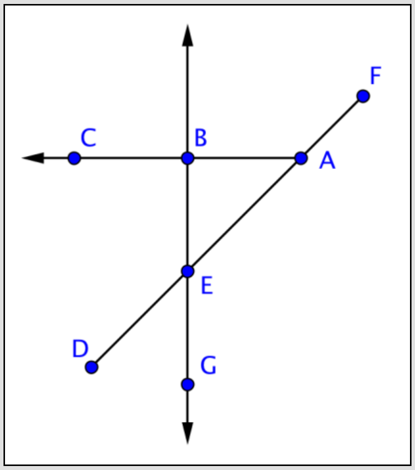
Summary

A system of linear equations is two or more linear equations that have the same variables. You can graph the equations as a system to find out whether the system has no solutions (represented by parallel lines), one solution (represented by intersecting lines), or an infinite number of solutions (represented by two superimposed lines). While graphing systems of equations is a useful technique, relying on graphs to identify a specific point of intersection is not always an accurate way to find a precise solution for a system of equations.

1.1.1 Self Check Solutions

Self Check A

Which of the following is *not* represented in the image below?



A) $\overleftrightarrow{BG}$
B) $\overrightarrow{BA}$
C) $\overline{DF}$
D) $\overrightarrow{AC}$

A) $\overleftrightarrow{BG}$

Incorrect. A line goes through points *B* and *G*, so $\overleftrightarrow{BG}$ is shown. $\overrightarrow{BA}$, is not shown in this image.

B) $\overrightarrow{BA}$

Correct. This image does not show any ray that begins at point *B* and goes through point *A*.

C) $\overline{DF}$

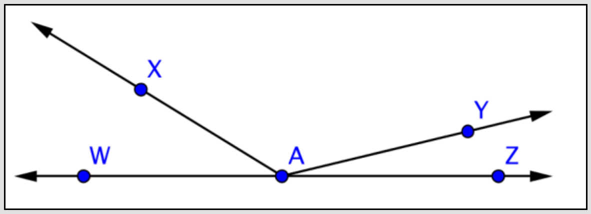
Incorrect. There is a line segment connecting points *D* and *F*, so $\overline{DF}$ is shown. $\overrightarrow{BA}$, is not shown in this image.

D) $\overrightarrow{AC}$

Incorrect. There is a ray beginning at point *A* and going through point *C*, so $\overrightarrow{AC}$ is shown. $\overrightarrow{BA}$, is not shown in this image.

Self Check B

Identify the acute angles in the image below.



A) $\angle WAX$, $\angle XAY$, and $\angle YAZ$
B) $\angle WAY$ and $\angle YAZ$
C) $\angle WAX$ and $\angle YAZ$
D) $\angle WAZ$ and $\angle XAY$

A) $\angle WAX$, $\angle XAY$, and $\angle YAZ$
Incorrect. $\angle WAX$ and $\angle YAZ$ are both acute angles, but $\angle XAY$ is an obtuse angle. So only $\angle WAX$ and $\angle YAZ$ are acute angles.

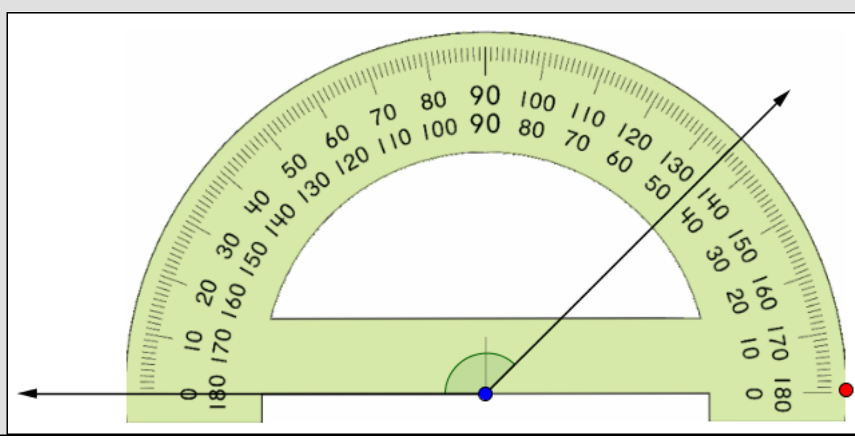
B) $\angle WAY$ and $\angle YAZ$
Incorrect. $\angle YAZ$ is an acute angle, but $\angle WAY$ is an obtuse angle. Both $\angle WAX$ and $\angle YAZ$ are acute angles.

C) $\angle WAX$ and $\angle YAZ$
Correct. Both $\angle WAX$ and $\angle YAZ$ are acute angles.

D) $\angle WAZ$ and $\angle XAY$
Incorrect. $\angle WAZ$ is a straight angle, and $\angle XAY$ is an obtuse angle. Both $\angle WAX$ and $\angle YAZ$ are acute angles.

Self Check C

What is the measurement of the angle shown below?



A) 45º
B) 135º
C) 145º
D) 180º

A) 45º
Incorrect. It looks like you started counting from the wrong side. In the picture above, notice how the bottom side of the angle is aligned with the 0º on the outside of the protractor. Continue to follow these numbers clockwise (10, 20, 30, …) until you get to the point where the other side of the angle crosses the protractor. The correct answer is 135º.

B) 135º
Correct. This protractor is aligned correctly, and the correct measurement is 135º.

C) 145º
Incorrect. It looks like you thought the side of the angle crossed the protractor between 140º and 150º; it actually crosses between 130º and 140º. The correct answer is 135º.

D) 180º
Incorrect. You looked at the wrong side of the angle. In the picture above, notice how the bottom side of the angle is aligned with the 0º on the outside of the protractor. Continue to follow these numbers clockwise (10, 20, 30, …) until you get to the point where the other side of the angle crosses the protractor. The correct answer is 135º.

1.1.2 Properties of Angles

Learning Objective(s)
1 Identify parallel and perpendicular lines.
2 Find measures of angles.
3 Identify complementary and supplementary angles.

Introduction

Imagine two separate and distinct lines on a plane. There are two possibilities for these lines: they will either intersect at one point, or they will never intersect. When two lines intersect, four angles are formed. Understanding how these angles relate to each other can help you figure out how to measure them, even if you only have information about the size of one angle.

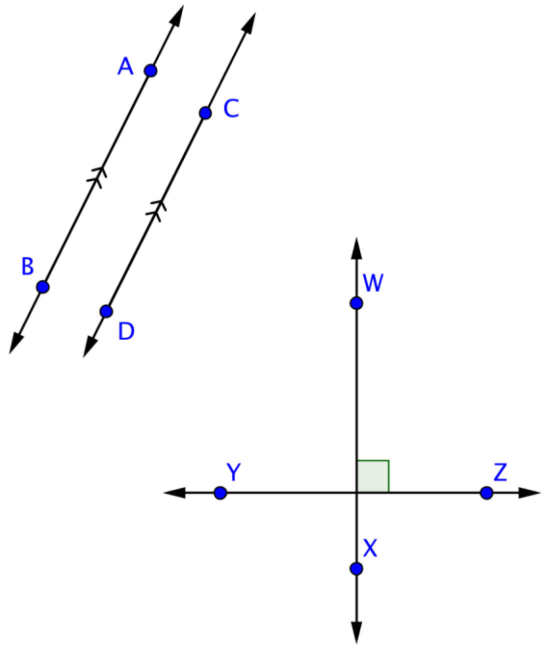
Parallel and Perpendicular

Objective **1**

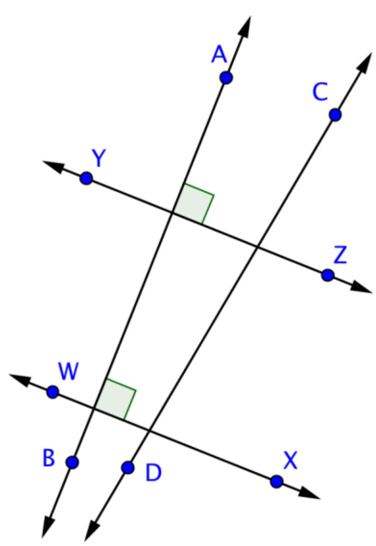
Parallel lines are two or more lines that never intersect. Likewise, parallel line segments are two line segments that never intersect even if the line segments were turned into lines that continued forever. Examples of parallel line segments are all around you, in the two sides of this page and in the shelves of a bookcase. When you see lines or structures that seem to run in the same direction, never cross one another, and are always the same distance apart, there's a good chance that they are parallel.

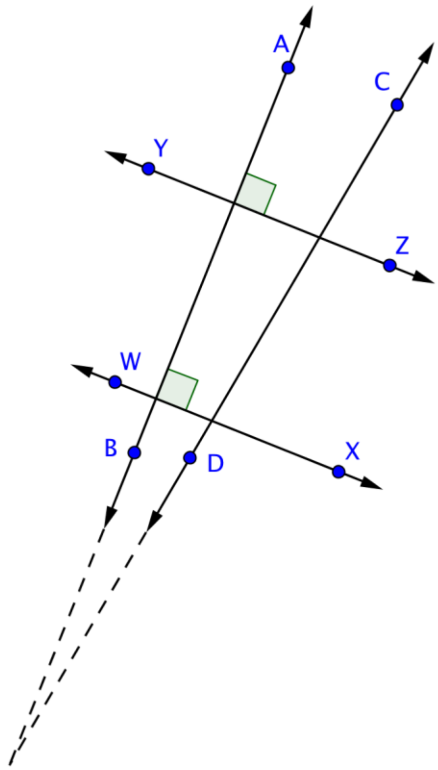
Perpendicular lines are two lines that intersect at a 90º (right) angle. And perpendicular line segments also intersect at a 90º (right) angle. You can see examples of perpendicular lines everywhere as well—on graph paper, in the crossing pattern of roads at an intersection, to the colored lines of a plaid shirt. In our daily lives, you may be happy to call two lines perpendicular if they merely seem to be at right angles to one another. When studying geometry, however, you need to make sure that two lines intersect at a 90º angle before declaring them to be perpendicular.

The image below shows some parallel and perpendicular lines. The geometric symbol for parallel is $||$, so you can show that $\overleftrightarrow{AB} \,||\, \overleftrightarrow{CD}$. Parallel lines are also often indicated by the marking >> on each line (or just a single > on each line). Perpendicular lines are indicated by the symbol $\perp$, so you can write $\overleftrightarrow{WX} \perp \overleftrightarrow{YZ}$.



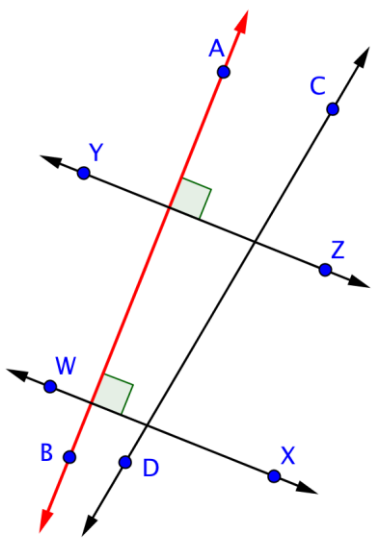
If two lines are parallel, then any line that is perpendicular to one line will also be perpendicular to the other line. Similarly, if two lines are both perpendicular to the same line, then those two lines are parallel to each other. Let's take a look at one example and identify some of these types of lines.

Example	
Problem	**Identify a set of parallel lines and a set of perpendicular lines in the image below.**
	

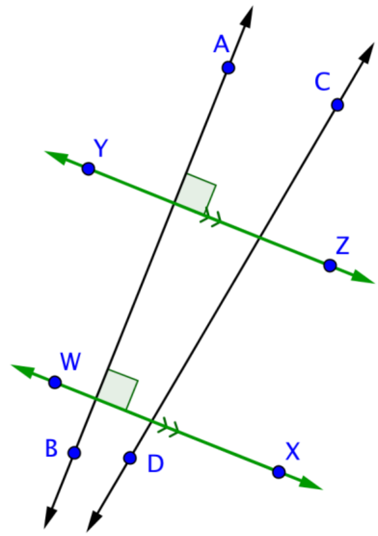


Parallel lines never meet, and perpendicular lines intersect at a right angle.

$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not intersect in this image, but if you imagine extending both lines, they will intersect soon. So, they are neither parallel nor perpendicular.



$\overleftrightarrow{AB}$ is perpendicular to both $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$, as indicated by the right angle marks at the intersection of those lines.



Since $\overleftrightarrow{AB}$ is perpendicular to both lines, then $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$ are parallel.

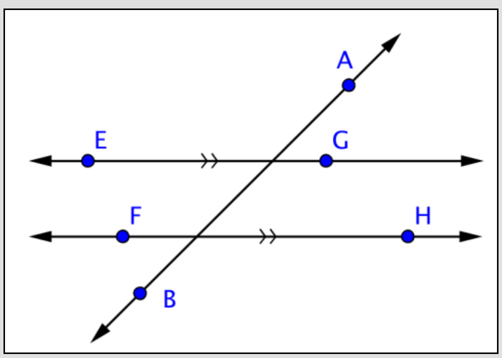
Answer

$\overleftrightarrow{WX} \parallel \overleftrightarrow{YZ}$

$\overleftrightarrow{AB} \perp \overleftrightarrow{WX}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{YZ}$

Self Check A

Which statement most accurately represents the image below?



A) $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

B) $\overleftrightarrow{AB} \perp \overleftrightarrow{EG}$

C) $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$

D) $\overleftrightarrow{AB} \parallel \overleftrightarrow{FH}$

Finding Angle Measurements

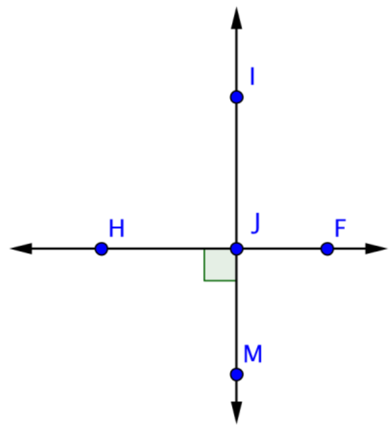
Objective **2**

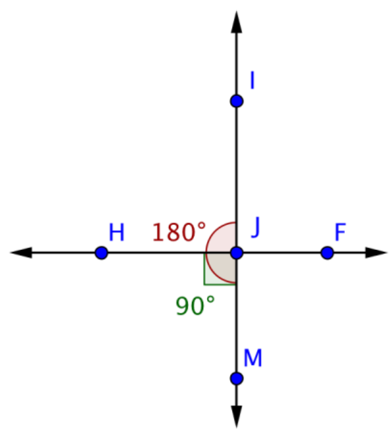
Understanding how parallel and perpendicular lines relate can help you figure out the measurements of some unknown angles. To start, all you need to remember is that perpendicular lines intersect at a 90º angle, and that a straight angle measures 180º.

The measure of an angle such as $\angle A$ is written as $m\angle A$. Look at the example below. How can you find the measurements of the unmarked angles?

Example

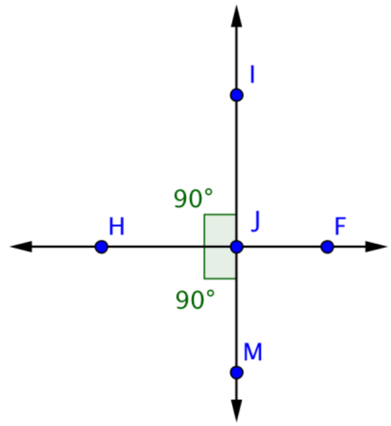
Problem **Find the measurement of $\angle IJF$.**





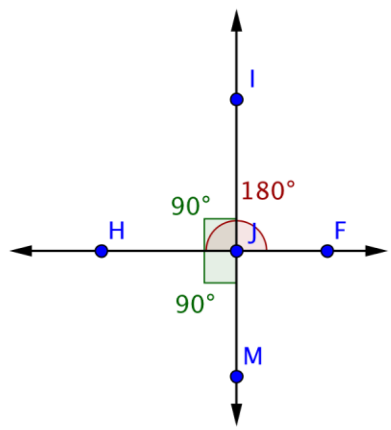
Only one angle, $\angle HJM$, is marked in the image. Notice that it is a right angle, so it measures 90º.

$\angle HJM$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$. Since $\overleftrightarrow{IM}$ is a line, $\angle IJM$ is a straight angle measuring 180º.



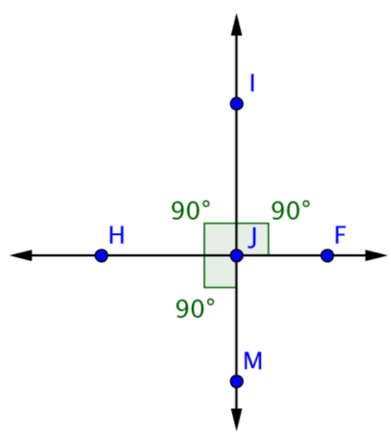
You can use this information to find the measurement of $\angle HJI$:

$$m\angle HJM + m\angle HJI = m\angle IJM$$
$$90^o + m\angle HJI = 180^o$$
$$m\angle HJI = 90^o$$



Now use the same logic to find the measurement of $\angle IJF$.

$\angle IJF$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$. Since $\overleftrightarrow{HF}$ is a line, $\angle HJF$ will be a straight angle measuring 180º.



You know that $\angle HJI$ measures 90º. Use this information to find the measurement of $\angle IJF$:

$$m\angle HJI + m\angle IJF = m\angle HJF$$
$$90 + m\angle IJF = 180^o$$
$$m\angle IJF = 90^o$$

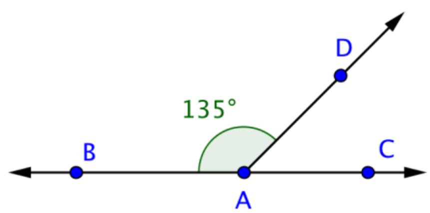
Answer $m\angle IJF = 90^o$

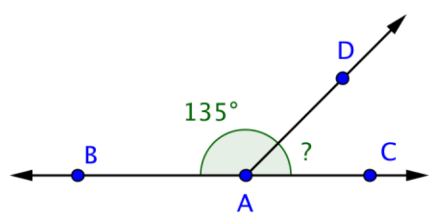
In this example, you may have noticed that angles $\angle HJI$, $\angle IJF$, and $\angle HJM$ are all right angles. (If you were asked to find the measurement of $\angle FJM$, you would find that angle to be 90º, too.) This is what happens when two lines are perpendicular—the four angles created by the intersection are all right angles.

Not all intersections happen at right angles, though. In the example below, notice how you can use the same technique as shown above (using straight angles) to find the measurement of a missing angle.

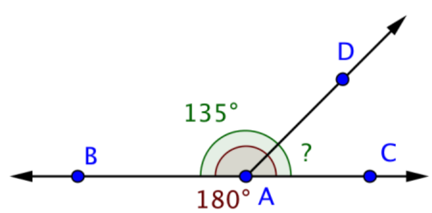
Example

Problem **Find the measurement of $\angle DAC$.**

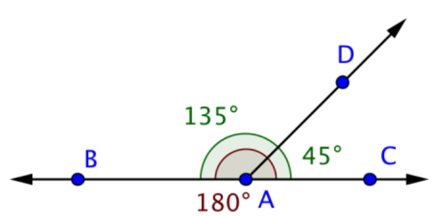




This image shows the line $\overleftrightarrow{BC}$ and the ray $\overrightarrow{AD}$ intersecting at point *A*. The measurement of $\angle BAD$ is 135º. You can use straight angles to find the measurement of $\angle DAC$.



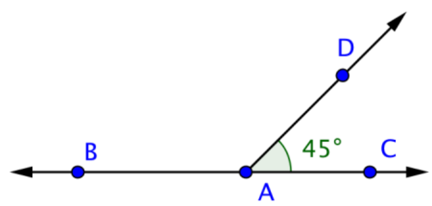
$\angle BAC$ is a straight angle, so it measures 180º.



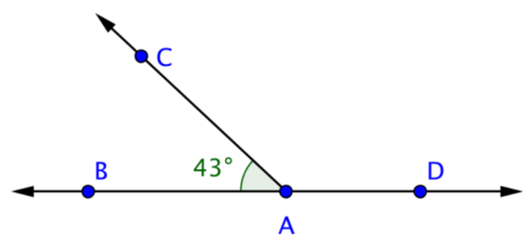
Use this information to find the measurement of $\angle DAC$.

$$m\angle BAD + m\angle DAC = m\angle BAC$$
$$135^\circ + m\angle DAC = 180^\circ$$
$$m\angle DAC = 45^\circ$$

Answer	$m\angle DAC = 45^{o}$



Self Check B
Find the measurement of $\angle CAD$.



A) 43º
B) 137º
C) 147º
D) 317º

Supplementary and Complementary

Objective **3**

In the example above, $m\angle BAC$ and $m\angle DAC$ add up to 180º. Two angles whose measures add up to 180º are called **supplementary angles**. There's also a term for two angles whose measurements add up to 90º, they are called **complementary angles**.

One way to remember the difference between the two terms is that *"c*orner" and "*c*omplementary" each begin with *c* (a 90º angle looks like a corner), while *s*traight and "*s*upplementary" each begin with *s* (a straight angle measures 180º).

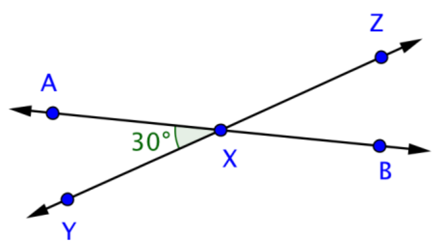
If you can identify supplementary or complementary angles within a problem, finding missing angle measurements is often simply a matter of adding or subtracting.

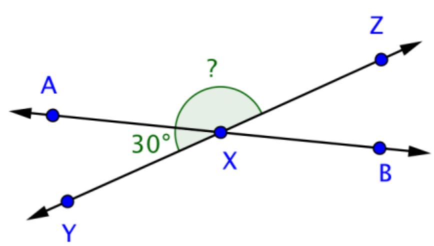
Example	
Problem	**Two angles are supplementary. If one of the angles measures 48º, what is the measurement of the other angle?**

	$m\angle A + m\angle B = 180^o$	Two supplementary angles make up a straight angle, so the measurements of the two angles will be 180º.
	$48^o + m\angle B = 180^o$ $m\angle B = 180^o - 48^o$ $m\angle B = 132^o$	You know the measurement of one angle. To find the measurement of the second angle, subtract 48º from 180º.
Answer	The measurement of the other angle is 132º.	

Example

Problem **Find the measurement of $\angle AXZ$.**

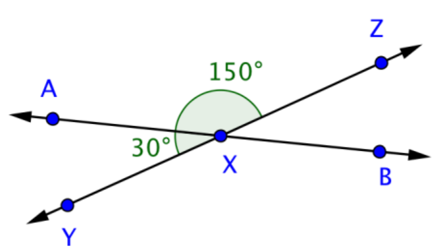




This image shows two intersecting lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{YZ}$. They intersect at point *X*, forming four angles.

Angles $\angle AXY$ and $\angle AXZ$ are supplementary because together they make up the straight angle $\angle YXZ$.

Use this information to find the measurement of $\angle AXZ$.

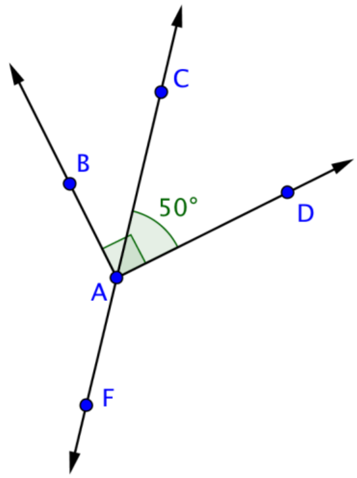


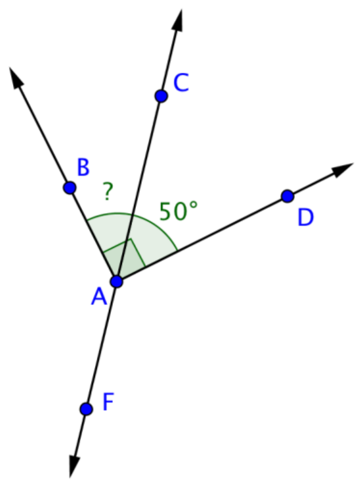
$$m\angle AXY + m\angle AXZ = m\angle YXZ$$
$$30^o + m\angle AXZ = 180^o$$
$$m\angle AXZ = 150^o$$

Answer $m\angle AXZ = 150^o$

Example

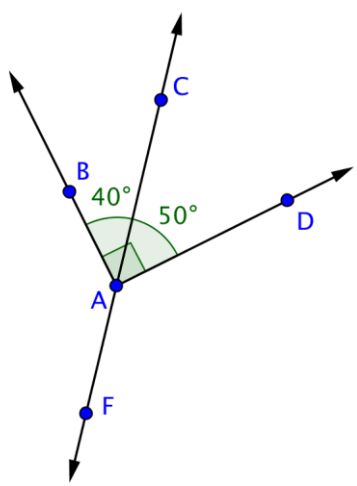
Problem **Find the measurement of $\angle BAC$.**





This image shows the line $\overleftrightarrow{CF}$ and the rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$, all intersecting at point *A*. Angle $\angle BAD$ is a right angle.

Angles $\angle BAC$ and $\angle CAD$ are complementary, because together they create $\angle BAD$.

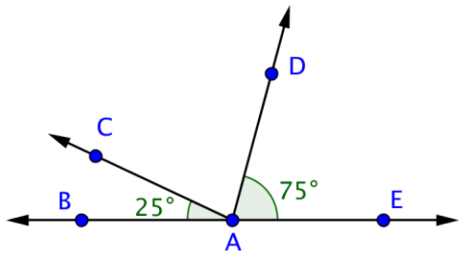


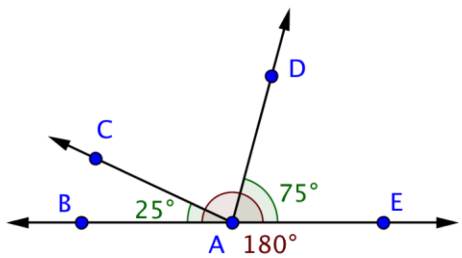
Use this information to find the measurement of $\angle BAC$.

$$m\angle BAC + m\angle CAD = m\angle BAD$$
$$m\angle BAC + 50^o = 90^o$$
$$m\angle BAC = 40^o$$

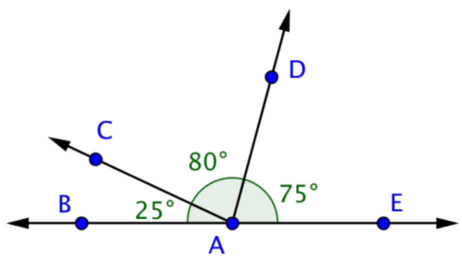
Answer $m\angle BAC = 40^o$

Example	
Problem	**Find the measurement of $\angle CAD$.**





You know the measurements of two angles here: $\angle CAB$ and $\angle DAE$. You also know that $m\angle BAE = 180^o$.



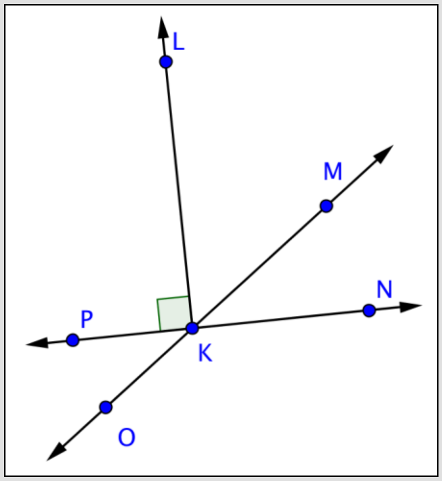
Use this information to find the measurement of $\angle CAD$.

$$m\angle BAC + m\angle CAD + m\angle DAE = m\angle BAE$$
$$25^o + m\angle CAD + 75^o = 180^o$$
$$m\angle CAD + 100^o = 180^o$$
$$m\angle CAD = 80^o$$

Answer $m\angle CAD = 80^o$

Self Check C

Which pair of angles is complementary?



A) $\angle PKO$ and $\angle MKN$
B) $\angle PKO$ and $\angle PKM$
C) $\angle LKP$ and $\angle LKN$
D) $\angle LKM$ and $\angle MKN$

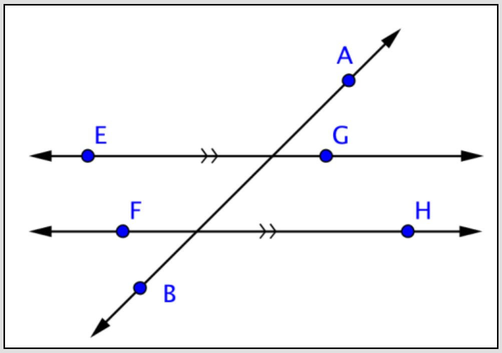
Summary

Parallel lines do not intersect, while perpendicular lines cross at a 90º angle. Two angles whose measurements add up to 180º are said to be supplementary, and two angles whose measurements add up to 90º are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

1.1.2 Self Check Solutions

Self Check A

Which statement most accurately represents the image below?



A) $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

B) $\overleftrightarrow{AB} \perp \overleftrightarrow{EG}$

C) $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$

D) $\overleftrightarrow{AB} \parallel \overleftrightarrow{FH}$

A) $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

Incorrect. This image shows the lines $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$, not $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel. The correct answer is $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$.

B) $\overleftrightarrow{AB} \perp \overleftrightarrow{EG}$

Incorrect. $\overleftrightarrow{AB}$ does intersect $\overleftrightarrow{EG}$, but the intersection does not form a right angle. This means that they cannot be perpendicular. The correct answer is $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$.

C) $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$

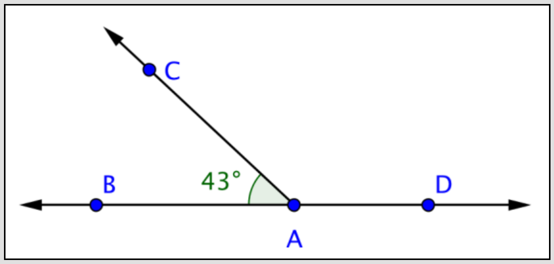
Correct. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel.

D) $\overleftrightarrow{AB} \parallel \overleftrightarrow{FH}$

Incorrect. $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$ intersect, so they cannot be parallel. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel. The correct answer is $\overleftrightarrow{FH} \parallel \overleftrightarrow{EG}$.

Self Check B

Find the measurement of $\angle CAD$.



A) 43º
B) 137º
C) 147º
D) 317º

A) 43º
Incorrect. You found the measurement of $\angle BAC$, not $\angle CAD$. Since $\angle BAC$ measures 43º, the measure of $\angle CAD$ must be 180º – 43º = 137º. The correct answer is 137º.

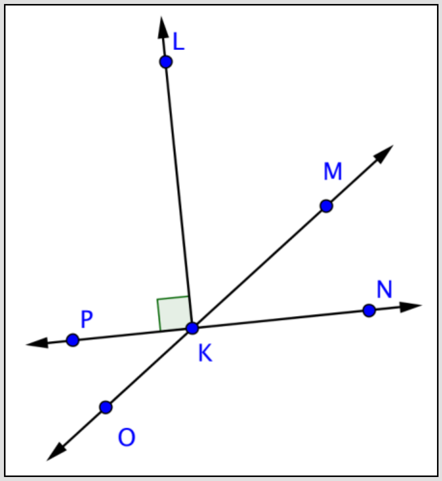
B) 137º
Correct. $\angle BAD$ is a straight angle measuring 180º. Since $\angle BAC$ measures 43º, the measure of $\angle CAD$ must be 180º – 43º = 137º.

C) 147º
Incorrect. It looks like you subtracted incorrectly. Since $\angle BAC$ measures 43º, $\angle CAD$ must be 180º – 43º = 137º. The correct answer is 137º.

D) 317º
Incorrect. Remember that a straight angle measures 180º, not 360º. Since $\angle BAC$ measures 43º, the measure of $\angle CAD$ must be 180º – 43º = 137º. The correct answer is 137º.

Self Check C

Which pair of angles is complementary?



A) $\angle PKO$ and $\angle MKN$
B) $\angle PKO$ and $\angle PKM$
C) $\angle LKP$ and $\angle LKN$
D) $\angle LKM$ and $\angle MKN$

A) $\angle PKO$ and $\angle MKN$

Incorrect. The measures of complementary angles add up to 90º. It looks like the measures of these angles may add up to 90º, but there is no way to be sure, so you cannot say that they are complementary. The correct answer is $\angle LKM$ and $\angle MKN$.

B) $\angle PKO$ and $\angle PKM$

Incorrect. $\angle PKO$ and $\angle PKM$ are supplementary angles (not complementary angles) because together they comprise the straight angle $\angle OKM$. The correct answer is $\angle LKM$ and $\angle MKN$.

C) $\angle LKP$ and $\angle LKN$

Incorrect. $\angle LKP$ and $\angle LKN$ are supplementary angles (not complementary angles) because together they comprise the straight angle $\angle PKN$. The correct answer is $\angle LKM$ and $\angle MKN$.

D) $\angle LKM$ and $\angle MKN$

Correct. The measurements of two complementary angles will add up to 90º. $\angle LKP$ is a right angle, so $\angle LKN$ must be a right angle as well. $\angle LKM + \angle MKN = \angle LKN$, so $\angle LKM$ and $\angle MKN$ are complementary.

1.1.3 Triangles

Learning Objective(s)
1 Identify equilateral, isosceles, scalene, acute, right, and obtuse triangles.
2 Identify whether triangles are similar, congruent, or neither.
3 Identify corresponding sides of congruent and similar triangles.
4 Find the missing measurements in a pair of similar triangles.
5 Solve application problems involving similar triangles.

Introduction

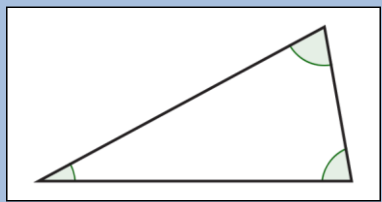
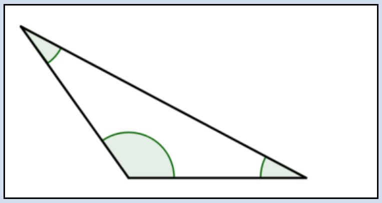
Geometric shapes, also called figures, are an important part of the study of geometry. The **triangle** is one of the basic shapes in geometry. It is the simplest shape within a classification of shapes called **polygons**. All triangles have three sides and three angles, but they come in many different shapes and sizes. Within the group of all triangles, the characteristics of a triangle's sides and angles are used to classify it even further. Triangles have some important characteristics, and understanding these characteristics allows you to apply the ideas in real-world problems.

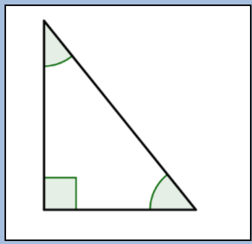
Classifying and Naming Triangles

Objective **1**

A polygon is a closed plane figure with three or more straight sides. Polygons each have a special name based on the number of sides they have. For example, the polygon with three sides is called a triangle because "tri" is a prefix that means "three." Its name also indicates that this polygon has three angles. The prefix "poly" means many.

The table below shows and describes three classifications of triangles. Notice how the types of angles in the triangle are used to classify the triangle.

Name of Triangle	Picture of Triangle	Description
Acute Triangle		A triangle with 3 acute angles (3 angles measuring between 0° and 90°).
Obtuse Triangle		A triangle with 1 obtuse angle (1 angle measuring between 90° and 180°).

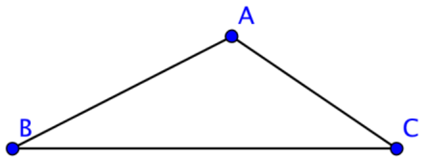
Right Triangle		A triangle containing one right angle (1 angle that measures 90°). Note that the right angle is shown with a corner mark and does not need to be labeled 90°.

The sum of the measures of the three interior angles of a triangle is always 180°. This fact can be applied to find the measure of the third angle of a triangle, if you are given the other two. Consider the examples below.

Example		
Problem	**A triangle has two angles that measure 35° and 75°. Find the measure of the third angle.**	
	35° + 75° + x = 180°	The sum of the three interior angles of a triangle is 180°.
	110º + x = 180º	Find the value of x.
	x = 180° – 110º x = 70°	
Answer	The third angle of the triangle measures 70°.	

Example		
Problem	**One of the angles in a right triangle measures 57º. Find the measurement of the third angle.**	
	57° + 90° + x = 180°	The sum of the three angles of a triangle is 180°. One of the angles has a measure of 90° as it is a right triangle.
	147º + x = 180°	Simplify.
	x = 180º - 147º x = 33 º	Find the value of x.
Answer	The third angle of the right triangle measures 33°.	

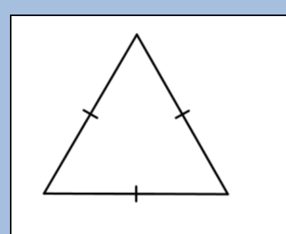
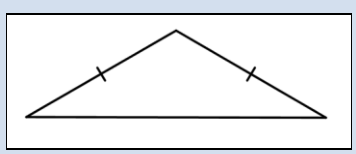
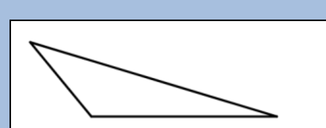
There is an established convention for naming triangles. The labels of the vertices of the triangle, which are generally capital letters, are used to name a triangle.



You can call this triangle *ABC* or ΔABC since *A*, *B*, and *C* are vertices of the triangle. When naming the triangle, you can begin with any vertex. Then keep the letters in order as you go around the polygon. The triangle above could be named in a variety of ways: ΔABC, or ΔCBA. The sides of the triangle are line segments *AB, AC,* and *CB.*

Just as triangles can be classified as acute, obtuse, or right based on their angles, they can also be classified by the length of their sides. Sides of equal length are called **congruent** sides. While we designate a segment joining points *A* and *B* by the notation $\overline{AB}$, we designate the length of a segment joining points *A* and *B* by the notation *AB* without a segment bar over it. The length *AB* is a number, and the segment $\overline{AB}$ is the collection of points that make up the segment.

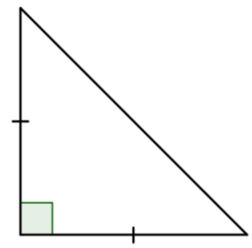
Mathematicians show congruency by putting a hash mark symbol through the middle of sides of equal length. If the hash mark is the same on one or more sides, then those sides are congruent. If the sides have different hash marks, they are *not* congruent. The table below shows the classification of triangles by their side lengths.

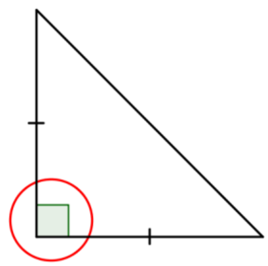
Name of Triangle	Picture of Triangle	Description
Equilateral Triangle		A triangle whose three sides have the same length. These sides of equal length are called congruent sides.
Isosceles Triangle		A triangle with exactly two congruent sides.
Scalene Triangle		A triangle in which all three sides are a different length.

To describe a triangle even more specifically, you can use information about both its sides and its angles. Consider this example.

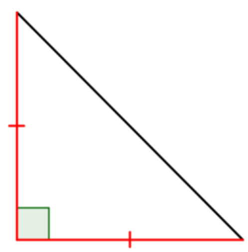
Example

Problem **Classify the triangle below.**





Notice what kind of angles the triangle has. Since one angle is a right angle, this is a right triangle.



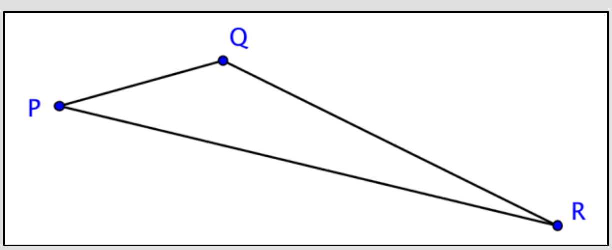
Notice the lengths of the sides. Are there congruence marks or other labels?

The congruence marks tell us there are two sides of equal length. So, this is an isosceles triangle.

Answer This is an isosceles right triangle.

Self Check A

Classify the triangle shown below.

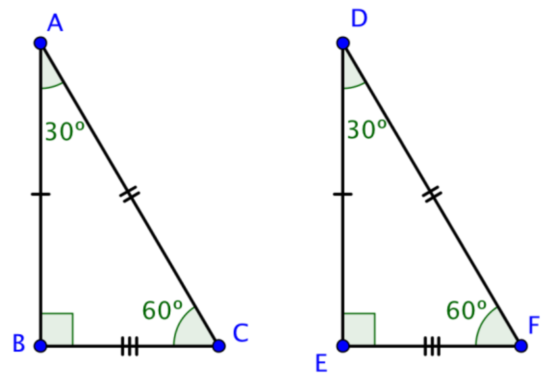


A) acute scalene
B) right isosceles
C) obtuse scalene
D) obtuse isosceles

Objective **2**

Identifying Congruent and Similar Triangles

Two triangles are congruent if they are exactly the same size and shape. In congruent triangles, the measures of **corresponding angles** and the lengths of **corresponding sides** are equal. Consider the two triangles shown below:



Since both $\angle B$ and $\angle E$ are right angles, these triangles are right triangles. Let's call these two triangles ΔABC and ΔDEF. These triangles are congruent if every pair of corresponding sides has equal lengths and every pair of corresponding angles has the same measure.

The corresponding sides are opposite the corresponding angles.

$\leftrightarrow$ means
"corresponds to"

$$\angle B \leftrightarrow \angle E$$

$$\angle A \leftrightarrow \angle D$$

$$\angle C \leftrightarrow \angle F$$

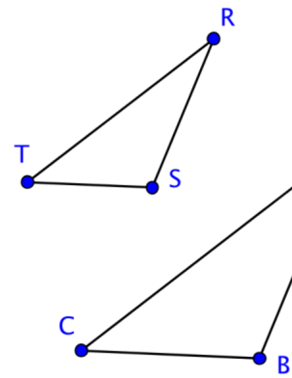
$$\overline{AB} \leftrightarrow \overline{DE}$$

$$\overline{AC} \leftrightarrow \overline{DF}$$

$$\overline{BC} \leftrightarrow \overline{EF}$$

ΔABC and ΔDEF are congruent triangles as the corresponding sides and corresponding angles are equal.

Let's take a look at another pair of triangles. Below are the triangles ΔABC and ΔRST.



These two triangles are surely not congruent because ΔRST is clearly smaller in size than ΔABC . But, even though they are not the same size, they do resemble one another. They are the same shape. The corresponding angles of these triangles look like they might have the same exact measurement, and if they did they would be congruent angles and we would call the triangles similar triangles.

Congruent angles are marked with hash marks, just as congruent sides are.

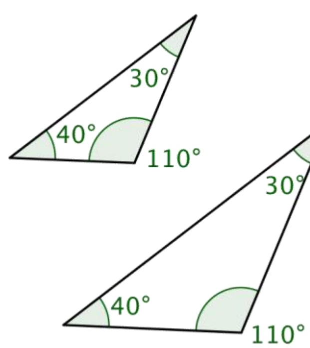


Image showing angle measurements of both triangles.

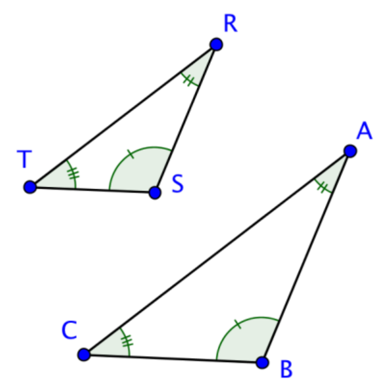


Image showing triangles ABC and RST using hash marks to show angle congruency.

We can also show congruent angles by using multiple bands within the angle, rather than multiple hash marks on one band. Below is an image using multiple bands within the angle.

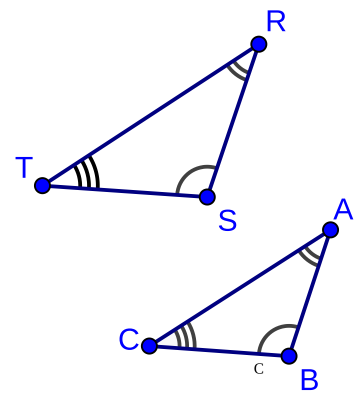


Image showing triangles ABC and RST using bands to show angle congruency.

If the corresponding angles of two triangles have the same measurements they are called **similar** triangles. This name makes sense because they have the same shape, but not necessarily the same size. When a pair of triangles is similar, the corresponding sides are proportional to one another. That means that there is a consistent scale factor that can be used to compare the corresponding sides. In the previous example, the side lengths of the larger triangle are all 1.4 times the length of the smaller. So, similar triangles are proportional to one another.

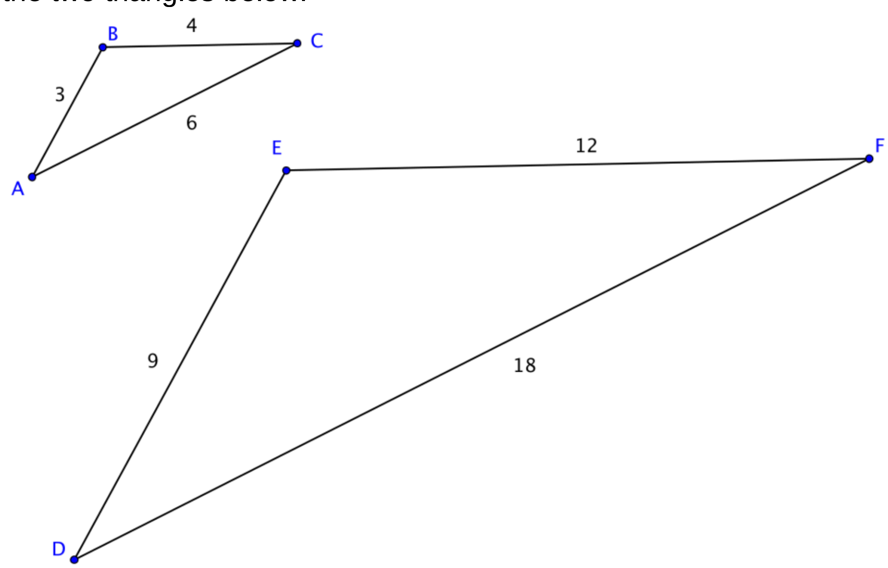
Just because two triangles *look* similar does not mean they *are* similar triangles in the mathematical sense of the word. Checking that the corresponding angles have equal measure is one way of being sure the triangles are similar.

Objective **3**

Corresponding Sides of Similar Triangles

There is another method for determining similarity of triangles that involves comparing the ratios of the lengths of the corresponding sides.

If the ratios of the pairs of corresponding sides are equal, the triangles are similar. Consider the two triangles below.



ΔABC is *not* congruent to ΔDEF because the side lengths of ΔDEF are longer than those of ΔABC. So, are these triangles similar? If they are, the corresponding sides should be proportional.

Since these triangles are oriented in the same way, you can pair the left, right, and bottom sides: $\overline{AB}$ and $\overline{DE}$, $\overline{BC}$ and $\overline{EF}$, $\overline{AC}$ and $\overline{DE}$. (You might call these the two shortest sides, the two longest sides, and the two leftover sides and arrived at the same ratios). Now we will look at the ratios of their lengths.

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Substituting the side length values into the proportion, you see that it is true:

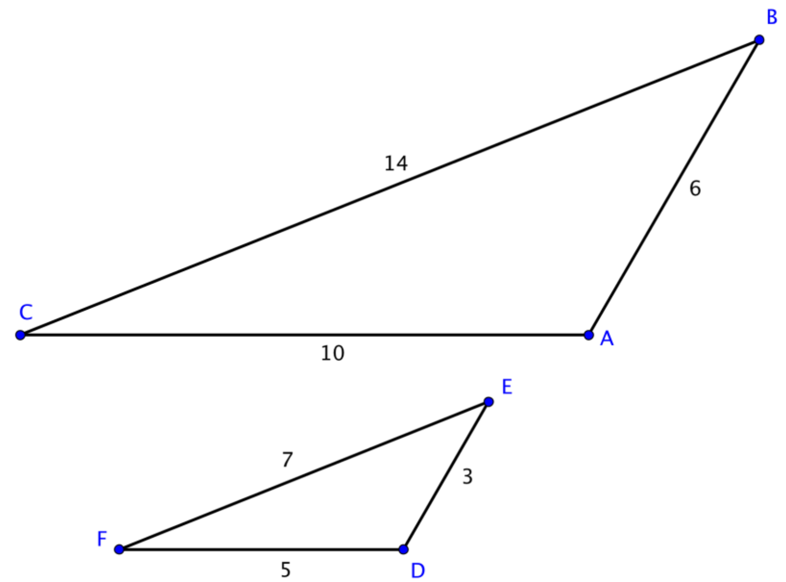
$$\frac{3}{9} = \frac{4}{12} = \frac{6}{18}$$

If the corresponding sides are proportional, then the triangles are similar. Triangles *ABC* and *DEF* are similar, but not congruent.

Let's use this idea of proportional corresponding sides to determine whether two more triangles are similar.

Example

Problem **Determine if the triangles below are similar by seeing if their corresponding sides are proportional.**

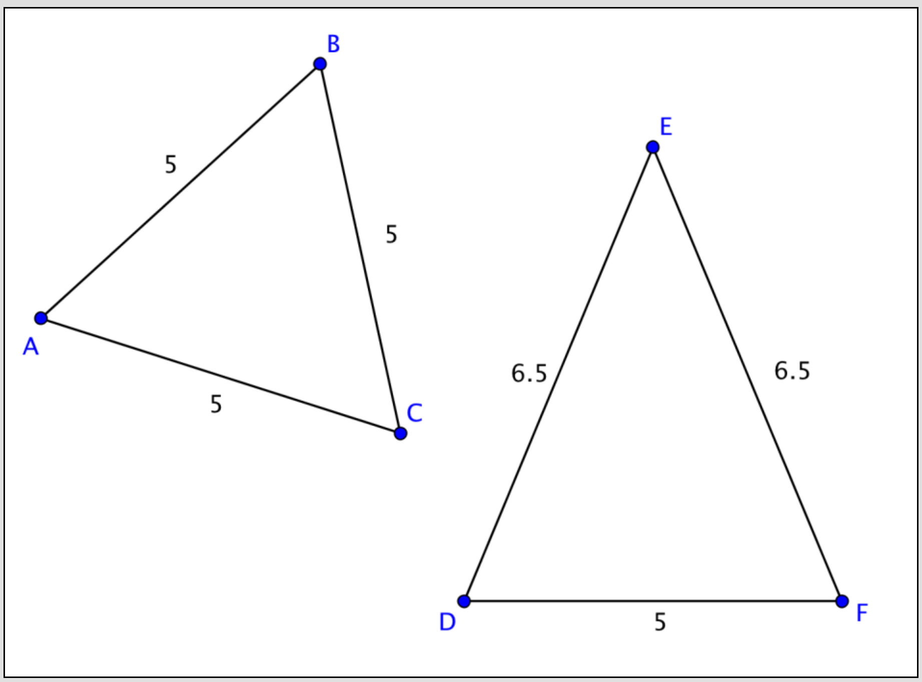


	$\overline{CA} \leftrightarrow \overline{FD}$ $\overline{AB} \leftrightarrow \overline{DE}$ $\overline{BC} \leftrightarrow \overline{EF}$	First determine the corresponding sides, which are opposite corresponding angles.
	$\frac{CA}{FD} = \frac{AB}{DE} = \frac{BC}{EF}$	Write the corresponding side lengths as ratios.
	$\frac{10}{5} = \frac{6}{3} = \frac{14}{7}$ $2 = 2 = 2$	Substitute the side lengths into the ratios, and determine if the ratios of the corresponding sides are equivalent. They are, so the triangles are similar.
Answer	ΔABC *and* ΔDEF are similar.	

The mathematical symbol ~ means "is similar to". So, you can write ΔABC is similar to ΔDEF as $\Delta ABC \sim \Delta DEF$.

Self Check B

Determine whether the two triangles are similar, congruent, or neither.



A) ΔABC and ΔDEF are congruent.
B) ΔABC and ΔDEF are similar.
C) ΔABC and ΔDEF are similar and congruent.
D) ΔABC and ΔDEF are neither similar nor congruent.

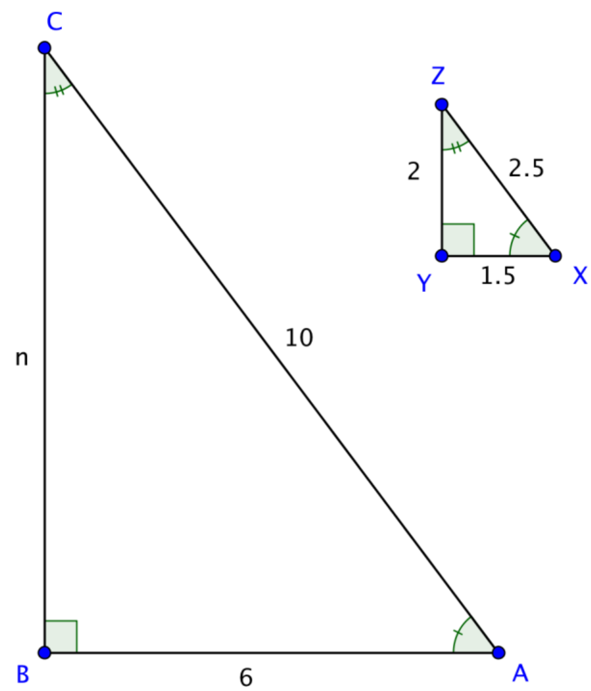
Objective **4**

Finding Missing Measurements in Similar Triangles

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let's look at an example.

Example

Problem **ΔABC and ΔXYZ are similar triangles. What is the length of side *BC*?**



$$\frac{BC}{YZ} = \frac{AB}{XY}$$

In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side.

$$\frac{n}{2} = \frac{6}{1.5}$$

Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be *n.*

$$2 \cdot 6 = 1.5 \cdot n$$

Solve for *n* using cross multiplication.

$$12 = 1.5n$$

$$8 = n$$

Answer The missing length of side *BC* is 8 units.

This process is fairly straightforward—but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

Objective **5**

Solving Application Problems Involving Similar Triangles

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

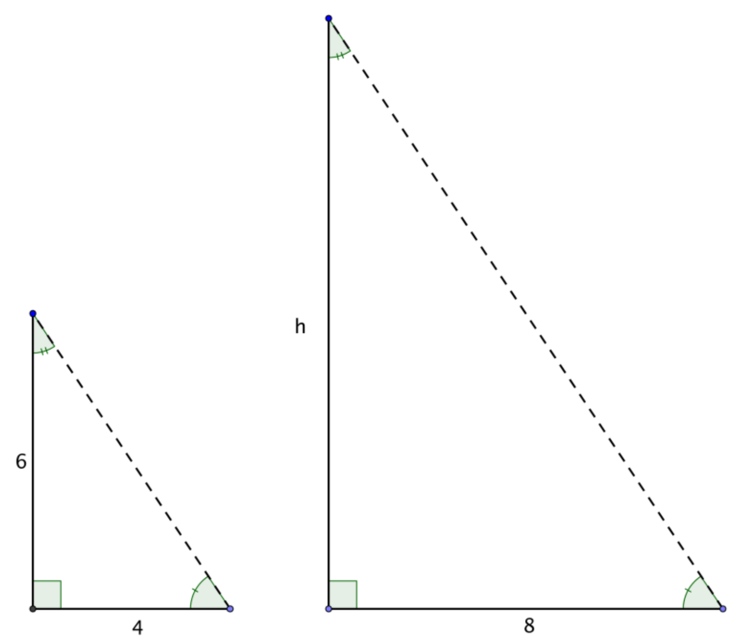
Let's consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let's figure out where the triangles are in this situation! The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

Example

Problem **When the sun is at a certain angle in the sky, a 6-foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?**



	$\frac{\text{Tree 1}}{\text{Tree 2}} = \frac{\text{Shadow 1}}{\text{Shadow 2}}$	The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side. Set up a proportion comparing the heights of the trees and the lengths of their shadows.
	$\frac{6}{h} = \frac{4}{8}$	Substitute in the known lengths. Call the missing tree height *h.*
	$6 \cdot 8 = 4h$ $48 = 4h$ $12 = h$	Solve for *h* using cross-multiplication.
Answer		The tree is 12 feet tall.

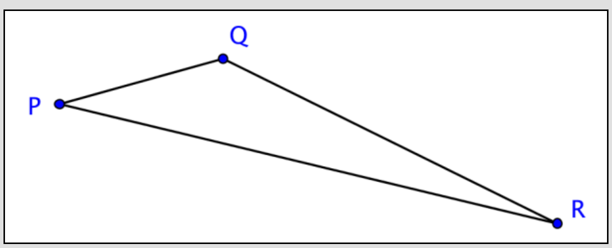
Summary

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180º. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

1.1.3 Self Check Solutions

Self Check A

Classify the triangle shown below.



A) acute scalene
B) right isosceles
C) obtuse scalene
D) obtuse isosceles

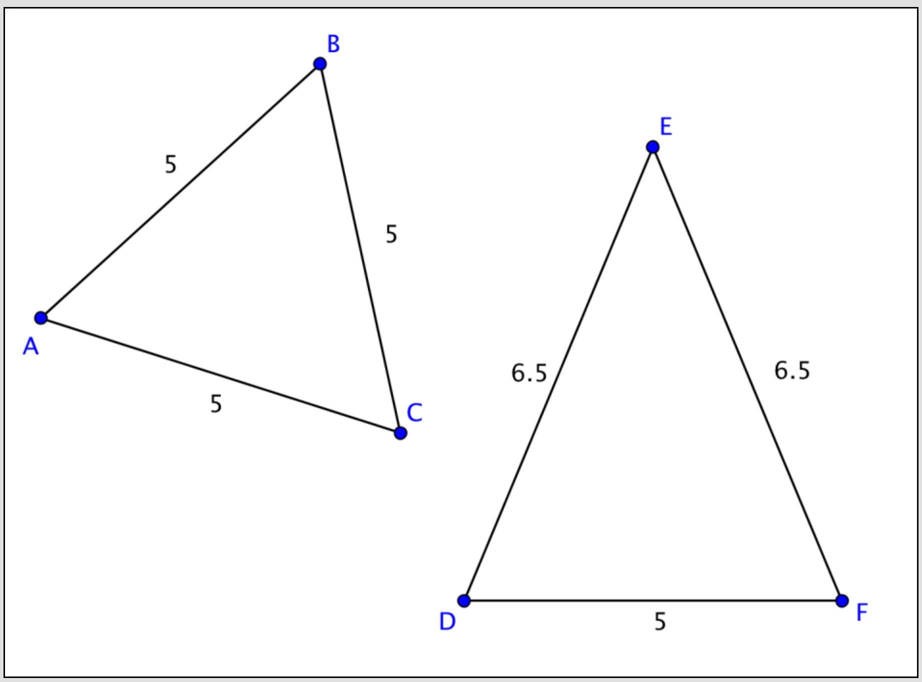
A) acute scalene
Incorrect. This triangle has one angle (angle Q) that is between 90º and 180º, so it is an obtuse triangle. It is also scalene because all the sides have different lengths. The correct answer is obtuse scalene.

B) right isosceles
Incorrect. This triangle does not contain a right angle. It has one angle (angle Q) that is between 90º and 180º, so it is an obtuse triangle. It is also scalene because all the sides have different lengths. The correct answer is obtuse scalene.

C) obtuse scalene
Correct. This triangle has vertices *P*, *Q*, and *R*, one angle (angle Q) that is between 90º and 180º, and sides of three different lengths.

D) obtuse isosceles
Incorrect. Although this triangle is obtuse, it does not have two sides of equal length. Its three sides are all different lengths, so it is scalene. The correct answer is obtuse scalene.

Self Check B
Determine whether the two triangles are similar, congruent, or neither.



A) ΔABC and ΔDEF are congruent.
B) ΔABC and ΔDEF are similar.
C) ΔABC and ΔDEF are similar and congruent.
D) ΔABC and ΔDEF are neither similar nor congruent.

A) ΔABC and ΔDEF are congruent.
Incorrect. Congruent triangles have corresponding sides of equal length and corresponding angles of equal measure. They are the same exact size and shape. ΔABC is equilateral and ΔDEF is isosceles, so they are not the same exact shape. The correct answer is ΔABC and ΔDEF are neither similar nor congruent.

B) ΔABC and ΔDEF are similar.
Incorrect. The ratios of the corresponding sides are not equal, so the triangles cannot be similar: $\frac{6.5}{5} = \frac{6.5}{5} \neq \frac{5}{5}$. The correct answer is ΔABC and ΔDEF are neither similar nor congruent.

C) ΔABC and ΔDEF are similar and congruent.
Incorrect. All congruent triangles are similar, but these triangles are not congruent. Congruent triangles have corresponding sides of equal length and corresponding angles of equal measure. ΔABC is equilateral and ΔDEF is isosceles, so they are not the same exact shape. The correct answer is ΔABC and ΔDEF are neither similar nor congruent.

D) ΔABC and ΔDEF are neither similar nor congruent.
Correct. The corresponding angle measures are not known to be equal as shown by the absence of congruence marks on the angles. Also, the ratios of the corresponding sides are not equal: $\frac{6.5}{5} = \frac{6.5}{5} \neq \frac{5}{5}$.

1.1.4 The Pythagorean Theorem

Learning Objective(s)
1 Use the Pythagorean Theorem to find the unknown side of a right triangle.
2 Solve application problems involving the Pythagorean Theorem.

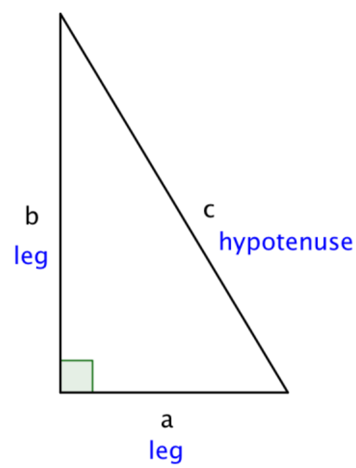
Introduction

A long time ago, a Greek mathematician named **Pythagoras** discovered an interesting property about **right triangles**: the sum of the squares of the lengths of each of the triangle's **legs** is the same as the square of the length of the triangle's **hypotenuse**. This property—which has many applications in science, art, engineering, and architecture—is now called the **Pythagorean Theorem**.

Let's take a look at how this theorem can help you learn more about the construction of triangles. And the best part—you don't even have to speak Greek to apply Pythagoras' discovery.

The Pythagorean Theorem

Pythagoras studied right triangles, and the relationships between the legs and the hypotenuse of a right triangle, before deriving his theory.



The Pythagorean Theorem
If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

This relationship is represented by the formula: $a^2 + b^2 = c^2$

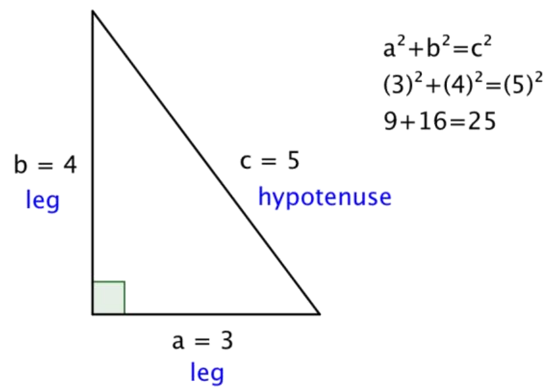
In the box above, you may have noticed the word "square," as well as the small 2s to the top right of the letters in $a^2 + b^2 = c^2$. To **square** a number means to multiply it by itself.

So, for example, to square the number 5 you multiply 5 • 5, and to square the number 12, you multiply 12 • 12. Some common squares are shown in the table below.

Number	**Number Times Itself**	**Square**
1	$1^2 = 1 \cdot 1$	1
2	$2^2 = 2 \cdot 2$	4
3	$3^2 = 3 \cdot 3$	9
4	$4^2 = 4 \cdot 4$	16
5	$5^2 = 5 \cdot 5$	25
10	$10^2 = 10 \cdot 10$	100

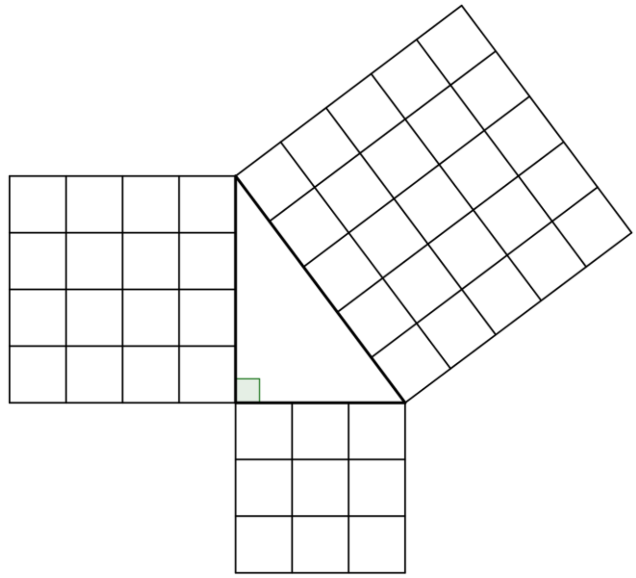
When you see the equation $a^2 + b^2 = c^2$, you can think of this as "the length of side *a* times itself, plus the length of side *b* times itself is the same as the length of side *c* times itself."

Let's try out all of the Pythagorean Theorem with an actual right triangle.



This theorem holds true for this right triangle—the sum of the squares of the lengths of both legs is the same as the square of the length of the hypotenuse. And, in fact, it holds true for all right triangles.

The Pythagorean Theorem can also be represented in terms of area. In any right triangle, the area of the square drawn from the hypotenuse is equal to the sum of the areas of the squares that are drawn from the two legs. You can see this illustrated below in the same 3-4-5 right triangle.

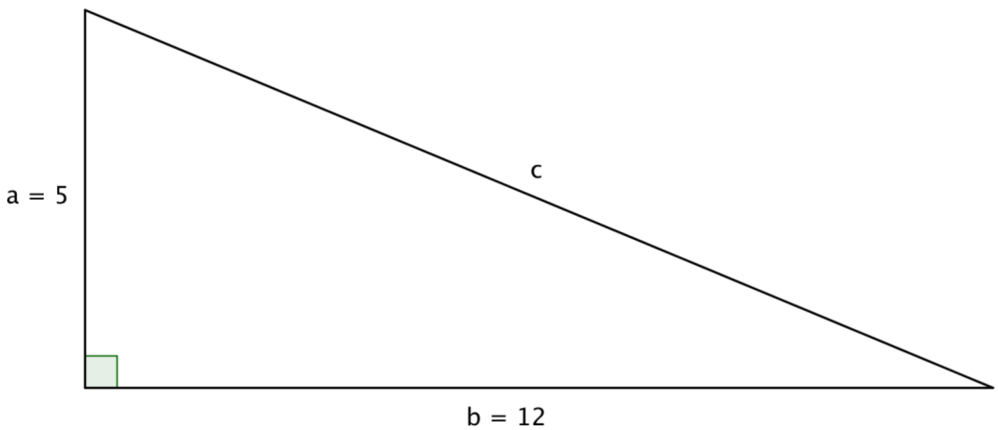


Note that the Pythagorean Theorem only works with *right* triangles.

Finding the Length of the Hypotenuse

Objective **1**

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle if you know the length of the triangle's other two sides, called the legs. Put another way, if you know the lengths of *a* and *b*, you can find *c*.



In the triangle above, you are given measures for legs *a* and *b*: 5 and 12, respectively. You can use the Pythagorean Theorem to find a value for the length of *c*, the hypotenuse.

$a^2+b^2=c^2$	The Pythagorean Theorem.
$(5)^2+(12)^2=c^2$	Substitute known values for *a* and *b*.
$25+144=c^2$	Evaluate.

$169 = c^2$	Simplify. To find the value of *c*, think about a number that, when multiplied by itself, equals 169. Does 10 work? How about 11? 12? 13? (You can use a calculator to multiply if the numbers are unfamiliar.)
13 = *c*	The square root of 169 is 13.

Using the formula, you find that the length of *c*, the hypotenuse, is 13.

In this case, you did not know the value of *c*—you were given the square of the length of the hypotenuse, and had to figure it out from there. When you are given an equation like $169 = c^2$ and are asked to find the value of *c*, this is called finding the **square root** of a number. (Notice you found a number, *c*, whose square was 169.)

Finding a square root takes some practice, but it also takes knowledge of multiplication, division, and a little bit of trial and error. Look at the table below.

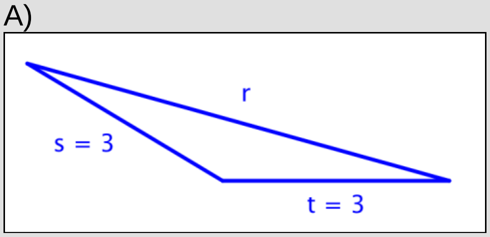
Number *x*	**Number *y* which, when multiplied by itself, equals number *x***	**Square root *y***
1	1 • 1	1
4	2 • 2	2
9	3 • 3	3
16	4 • 4	4
25	5 • 5	5
100	10 • 10	10

It is a good habit to become familiar with the squares of the numbers from 0–10, as these arise frequently in mathematics. If you can remember those square numbers—or if you can use a calculator to find them—then finding many common square roots will be just a matter of recall.

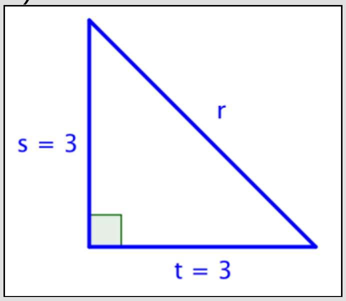
Self Check A

For which of these triangles is $(3)^2 + (3)^2 = r^2$?

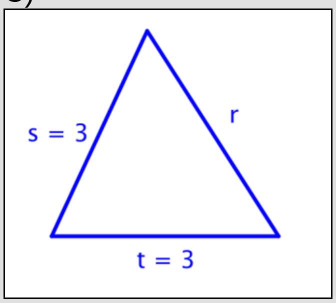
A)



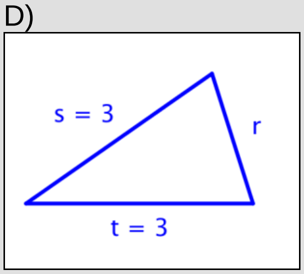
B)



C)



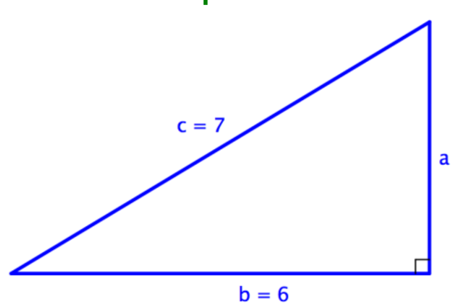
D)



Finding the Length of a Leg

You can use the same formula to find the length of a right triangle's leg if you are given measurements for the lengths of the hypotenuse and the other leg. Consider the example below.

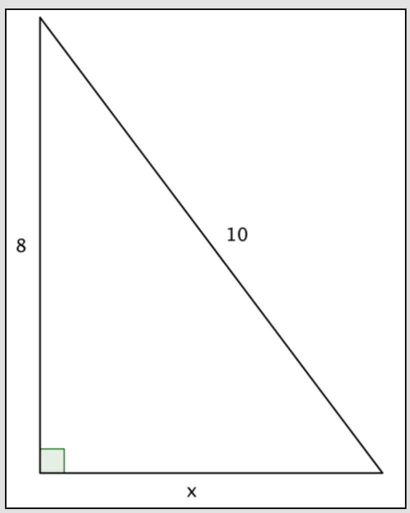
Example	
Problem	**Find the length of side *a* in the triangle below. Use a calculator to estimate the square root to one decimal place.**



	a = ? b = 6 c = 7	In this right triangle, you are given the measurements for the hypotenuse, *c*, and one leg, *b*. The hypotenuse is always opposite the right angle and it is always the longest side of the triangle.
	$a^2 + b^2 = c^2$ $a^2 + 6^2 = 7^2$	To find the length of leg *a*, substitute the known values into the Pythagorean Theorem.
	$a^2 + 36 = 49$ $a^2 = 13$	Solve for a^2. Think: what number, when added to 36, gives you 49?
	$a \approx 3.6$	Use a calculator to find the square root of 13. The calculator gives an answer of 3.6055…, which you can round to 3.6. (Since you are approximating, you use the symbol $\approx$.)
Answer	$a \approx 3.6$	

Self Check B

Which of the following correctly uses the Pythagorean Theorem to find the missing side, *x?*



A) $8^2 + 10^2 = x^2$

B) x + 8 = 10

C) $x^2 + 8^2 = 10^2$

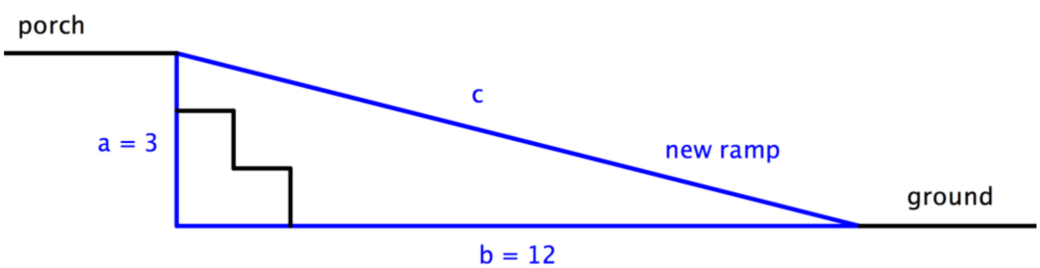
D) $x^2 + 10^2 = 8^2$

Using the Theorem to Solve Real World Problems

Objective **2**

The Pythagorean Theorem is perhaps one of the most useful formulas you will learn in mathematics because there are so many applications of it in real world settings. Architects and engineers use this formula extensively when building ramps, bridges, and buildings. Look at the following examples.

Example	
Problem	**The owners of a house want to convert a stairway leading from the ground to their back porch into a ramp. The porch is 3 feet off the ground, and due to building regulations the ramp must start 12 feet away from the base of the porch. How long will the ramp be?** **Use a calculator to find the square root, and round the answer to the nearest tenth.**
	To solve a problem like this one, it often makes sense to draw a simple diagram showing where the legs and hypotenuse of the triangle lie.

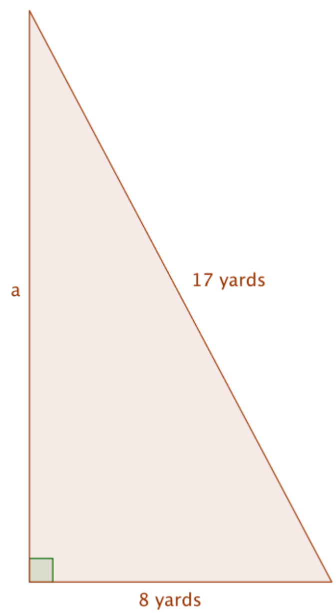


$a = 3$ $b = 12$ $c = ?$	Identify the legs and the hypotenuse of the triangle. You know that the triangle is a *right* triangle since the ground and the raised portion of the porch are perpendicular—this means you can use the Pythagorean Theorem to solve this problem. Identify *a*, *b*, and *c*.
$a^2 + b^2 = c^2$ $3^2 + 12^2 = c^2$ $9 + 144 = c^2$ $153 = c^2$	Use the Pythagorean Theorem to find the length of *c*.
$12.4 = c$	Use a calculator to find *c*. The square root of 153 is 12.369…, so you can round that to 12.4.

Answer The ramp will be 12.4 feet long.

Example

Problem	**A sailboat has a large sail in the shape of a right triangle. The longest edge of the sail measures 17 yards, and the bottom edge of the sail is 8 yards. How tall is the sail?**



Draw an image to help you visualize the problem. In a right triangle, the hypotenuse will always be the longest side, so here it must be 17 yards. The problem also tells you that the bottom edge of the triangle is 8 yards.

$$a^2 + b^2 = c^2$$

Setup the Pythagorean Theorem.

$$a^2 + 8^2 = 17^2$$

$$a^2 + 64 = 289$$

$$a^2 = 225$$

$$a = 15$$

15 • 15 = 225, so a = 15.

Answer The height of the sail is 15 yards.

Summary

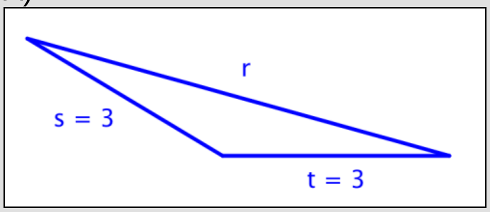
The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle's legs is the same as the square of the length of the triangle's hypotenuse. This theorem is represented by the formula $a^2 + b^2 = c^2$. Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

1.1.4 Self Check Solutions

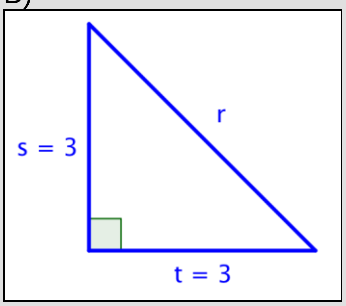
Self Check A

For which of these triangles is $(3)^2 + (3)^2 = r^2$?

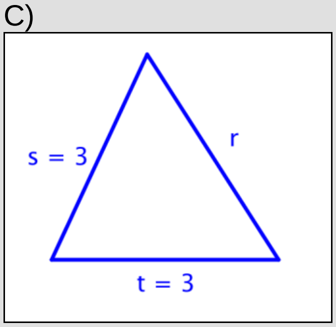
A)



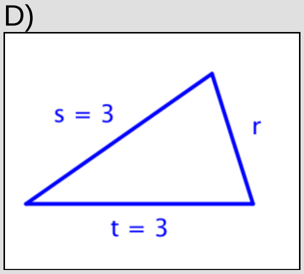
B)



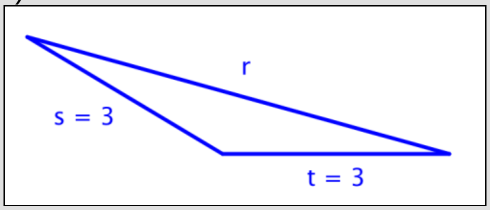
C)



D)

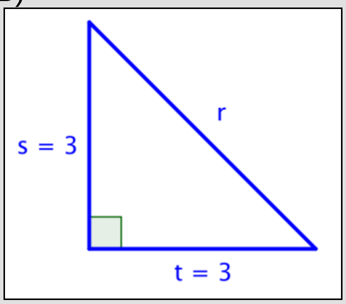


A)



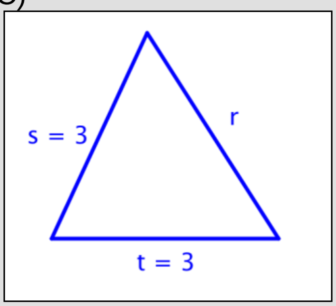
Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

B)



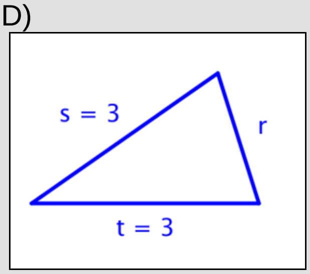
Correct. This is a right triangle; when you sum the squares of the lengths of the sides, you get the square of the length of the hypotenuse.

C)



Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

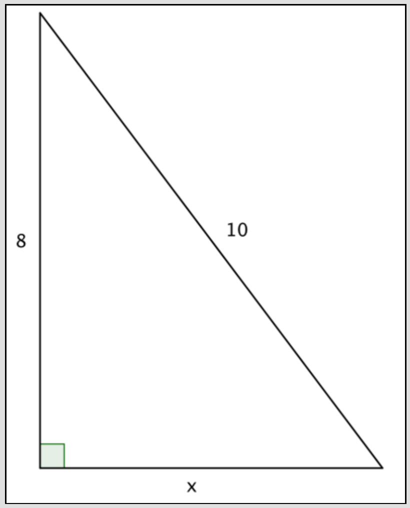
D)



Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

Self Check B

Which of the following correctly uses the Pythagorean Theorem to find the missing side, *x?*



A) $8^2 + 10^2 = x^2$

B) $x + 8 = 10$

C) $x^2 + 8^2 = 10^2$

D) $x^2 + 10^2 = 8^2$

A) $8^2 + 10^2 = x^2$
Incorrect. In this triangle, you know the hypotenuse (the side opposite the right angle) has a length of 10. The lengths of the legs are 8 and *x*. The correct answer is $x^2 + 8^2 = 10^2$.

B) $x + 8 = 10$
Incorrect. The Pythagorean Theorem is a relationship between the lengths of the sides squared. The correct answer is $x^2 + 8^2 = 10^2$.

C) $x^2 + 8^2 = 10^2$
Correct. In this triangle, the hypotenuse has length 10, and the legs have length 8 and *x*. Substituting into the Pythagorean Theorem you have: $x^2 + 8^2 = 10^2$; this equation is the same as $x^2 + 64 = 100$, or $x^2 = 36$. What number, times itself, equals 36? That would make $x = 6$.

D) $x^2 + 10^2 = 8^2$
Incorrect. In this triangle, the hypotenuse has length 10 (always the longest side of the triangle and the side opposite the right angle) not 8. The correct answer is $x^2 + 8^2 = 10^2$.

1.2.1 Quadrilaterals

Learning Objective(s)
1 Identify properties, including angle measurements, of quadrilaterals.

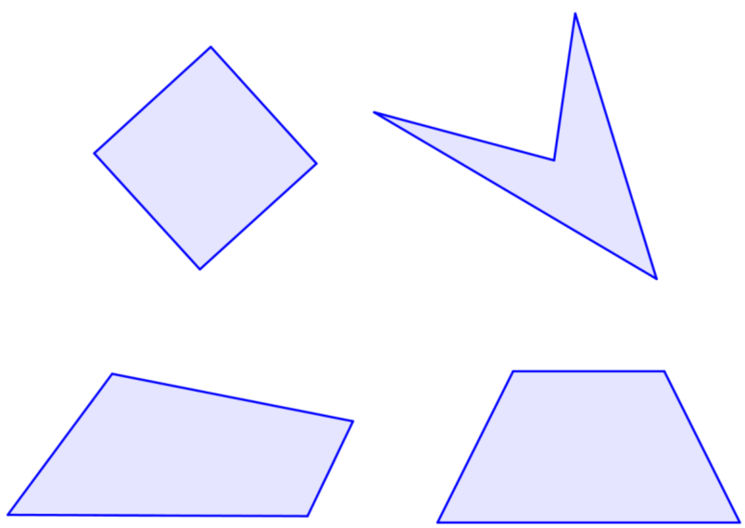
Introduction

Quadrilaterals are a special type of polygon. As with triangles and other polygons, quadrilaterals have special properties and can be classified by characteristics of their angles and sides. Understanding the properties of different quadrilaterals can help you in solving problems that involve this type of polygon.

Defining a Quadrilateral

Picking apart the name "quadrilateral" helps you understand what it refers to. The prefix "quad-" means "four," and "lateral" is derived from the Latin word for "side." So a quadrilateral is a four-sided polygon.

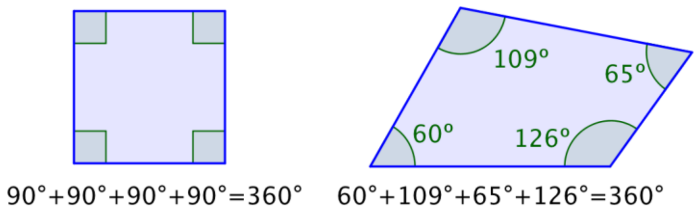
Since it is a **polygon**, you know that it is a two-dimensional figure made up of straight sides. A quadrilateral also has four angles formed by its four sides. Below are some examples of quadrilaterals. Notice that each figure has four straight sides and four angles.



Objective **1**

Interior Angles of a Quadrilateral

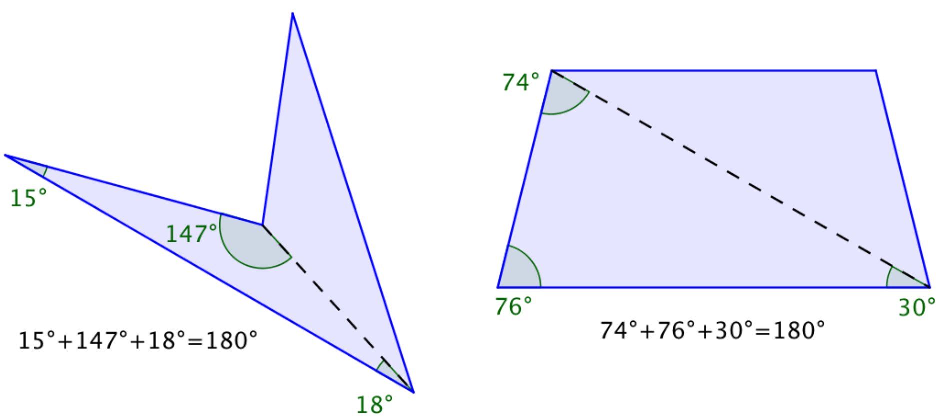
The sum of the interior angles of any quadrilateral is 360°. Consider the two examples below.



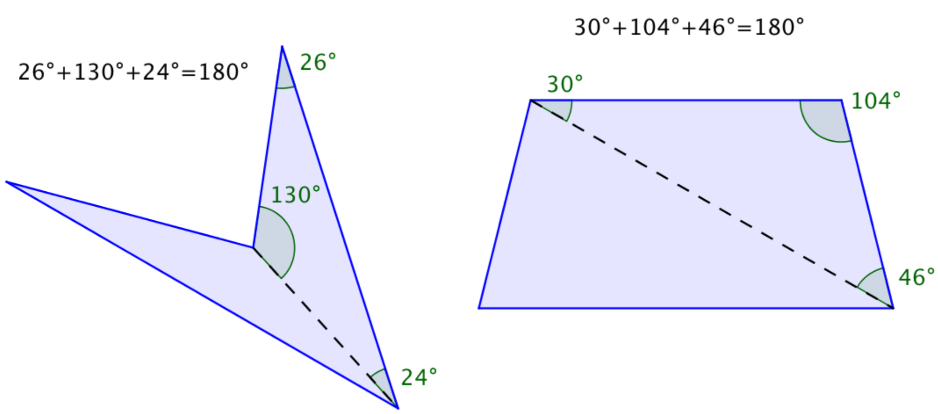
You could draw many quadrilaterals such as these and carefully measure the four angles. You would find that for every quadrilateral, the sum of the interior angles will always be 360°.

You can also use your knowledge of triangles as a way to understand why the sum of the interior angles of any quadrilateral is 360°. Any quadrilateral can be divided into two triangles as shown in the images below.

In the first image, the quadrilaterals have each been divided into two triangles. The angle measurements of one triangle are shown for each.



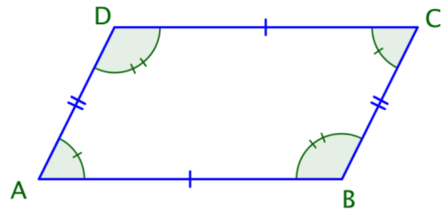
These measurements add up to 180º. Now look at the measurements for the other triangles—they also add up to 180º!



Since the sum of the interior angles of any triangle is 180° and there are two triangles in a quadrilateral, the sum of the angles for each quadrilateral is 360°.

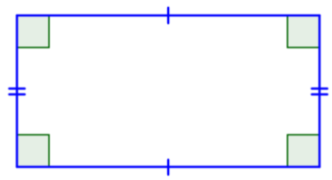
Specific Types of Quadrilaterals

Let's start by examining the group of quadrilaterals that have two pairs of parallel sides. These quadrilaterals are called **parallelograms**. They take a variety of shapes, but one classic example is shown below.

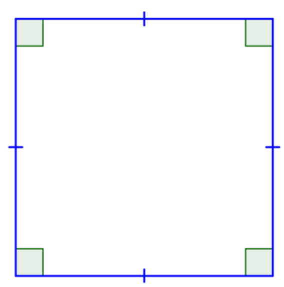


Imagine extending the pairs of opposite sides. They would never intersect because they are parallel. Notice, also, that the opposite angles of a parallelogram are congruent, as are the opposite sides. (Remember that "congruent" means "the same size.") The geometric symbol for congruent is $\cong$, so you can write $\angle A \cong \angle C$ and $\angle B \cong \angle D$. The parallel sides are also the same length: $\overline{AB} \cong \overline{DC}$ and $\overline{BC} \cong \overline{AD}$. These relationships are true for all parallelograms.

There are two special cases of parallelograms that will be familiar to you from your earliest experiences with geometric shapes. The first special case is called a **rectangle**. By definition, a rectangle is a parallelogram because its pairs of opposite sides are parallel. A rectangle also has the special characteristic that all of its angles are right angles; all four of its angles are congruent.

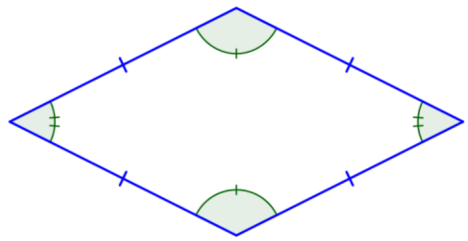


The other special case of a parallelogram is a special type of rectangle, a **square**. A square is one of the most basic geometric shapes. It is a special case of a parallelogram that has four congruent sides and four right angles.



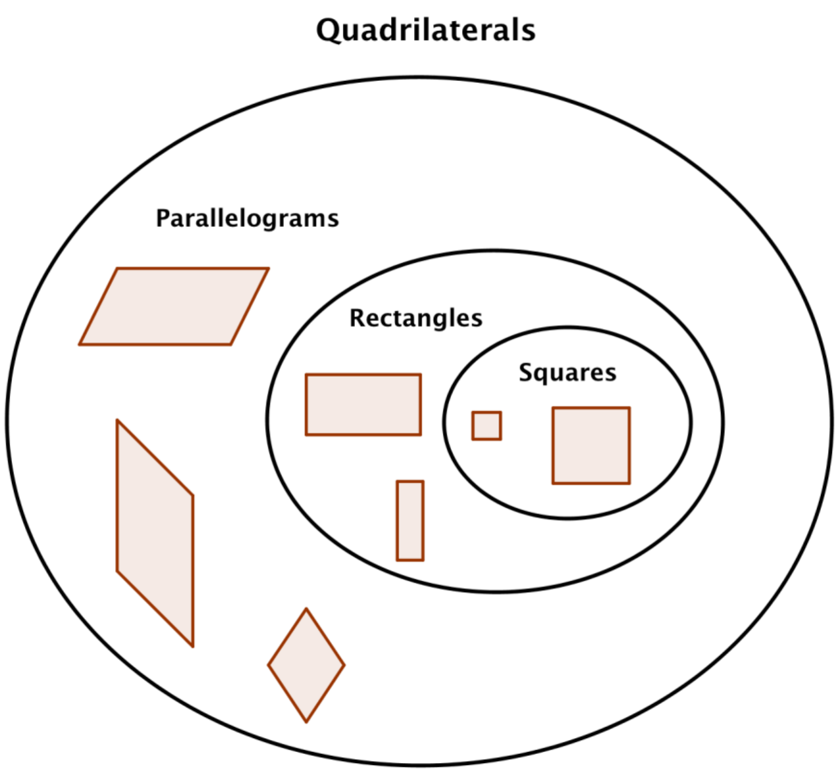
A square is also a rectangle because it has two sets of parallel sides and four right angles. A square is also a parallelogram because its opposite sides are parallel. So, a square can be classified in any of these three ways, with “parallelogram” being the least specific description and “square,” the most descriptive.

Another quadrilateral that you might see is called a **rhombus**. All four sides of a rhombus are congruent. Its properties include that each pair of opposite sides is parallel, also making it a parallelogram.

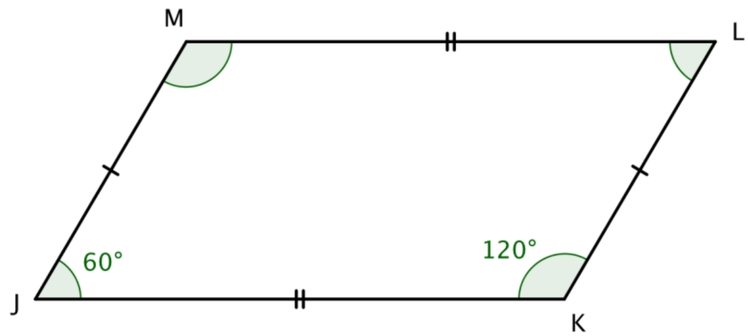


In summary, all squares are rectangles, but not all rectangles are squares. All rectangles are parallelograms, but not all parallelograms are rectangles. And *all* of these shapes are quadrilaterals.

The diagram below illustrates the relationship between the different types of quadrilaterals.

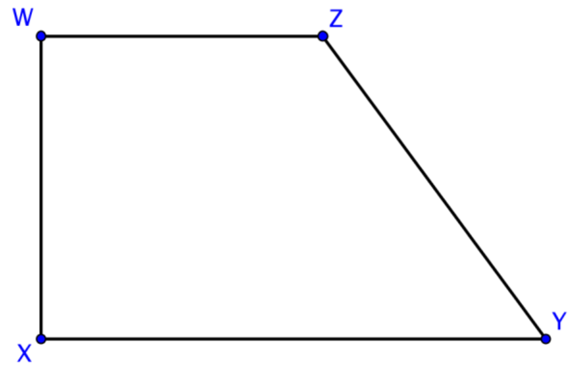


You can use the properties of parallelograms to solve problems. Consider the example that follows.

Example		
Problem	**Determine the measures of $\angle M$ and $\angle L$.**	
		
	$\angle L$ is opposite $\angle J$ $\angle M$ is opposite $\angle K$	Identify opposite angles.
	$\angle L \cong \angle J$ $\angle M \cong \angle K$	A property of parallelograms is that opposite angles are congruent.
	$m\angle J = 60°$, so $m\angle L = 60°$ $m\angle K = 120°$, so $m\angle M = 120°$	Use the given angle measurements to determine measures of opposite angles.
Answer	$m\angle L = 60°$ and $m\angle M = 120°$	

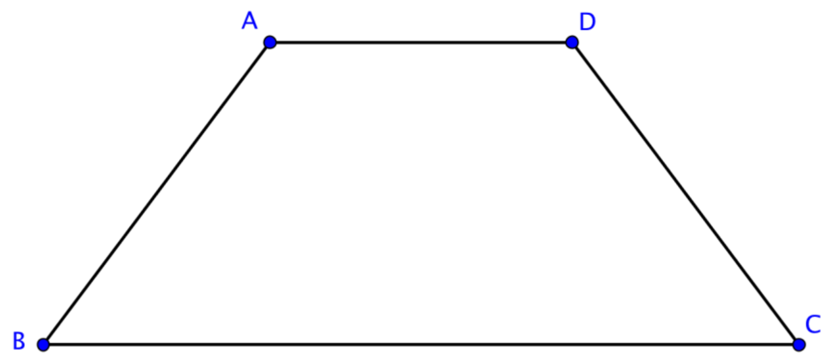
Trapezoids

There is another special type of quadrilateral. This quadrilateral has the property of having *only one* pair of opposite sides that are parallel. Here is one example of a **trapezoid**.



Notice that $\overline{XY} \parallel \overline{WZ}$, and that $\overline{WX}$ and $\overline{ZY}$ are not parallel. You can easily imagine that if you extended sides $\overline{WX}$ and $\overline{ZY}$, they would intersect above the figure.

If the non-parallel sides of a trapezoid are congruent, the trapezoid is called an **isosceles trapezoid**. Like the similarly named triangle that has two sides of equal length, the isosceles trapezoid has a pair of opposite sides of equal length. The other pair of opposite sides is parallel. Below is an example of an isosceles trapezoid.

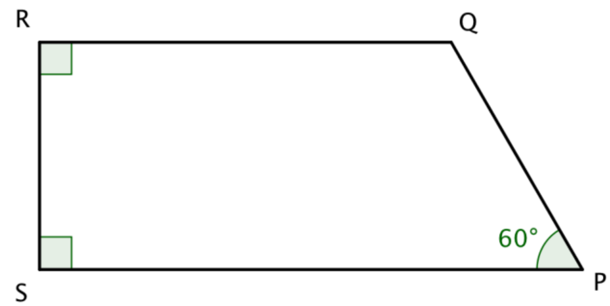


In this trapezoid *ABCD*, $\overline{BC} \parallel \overline{AD}$ and $\overline{AB} \cong \overline{CD}$.

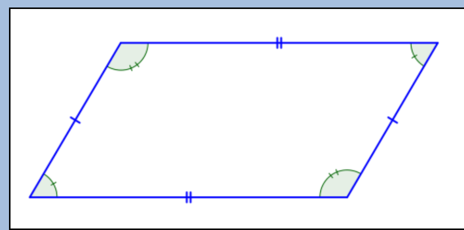
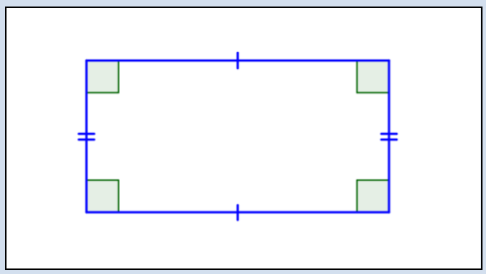
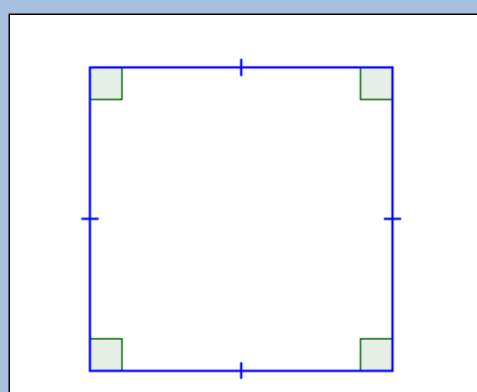
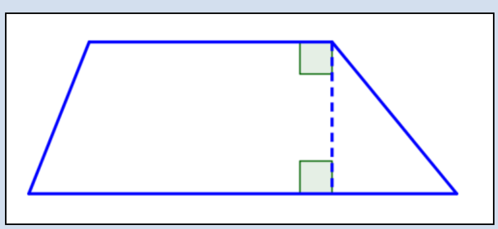
Self Check A
Which of the following statements is true?

A) Some trapezoids are parallelograms.
B) All trapezoids are quadrilaterals.
C) All rectangles are squares.
D) A shape cannot be a parallelogram and a quadrilateral.

You can use the properties of quadrilaterals to solve problems involving trapezoids. Consider the example below.

Example		
Problem	**Find the measure of $\angle Q$.**	
		
	$m\angle P + m\angle Q + m\angle R + m\angle S = 360°$	The sum of the measures of the interior angles of a quadrilateral is 360°.
	$m\angle R = 90°$ $m\angle S = 90°$	The square symbol indicates a right angle.
	$60° + m\angle Q + 90° + 90° = 360°$	Since three of the four angle measures are given, you can find the fourth angle measurement.
	$m\angle Q + 240° = 360°$ $m\angle Q = 120°$	Calculate the measurement of $\angle Q$. From the image, you can see that it is an obtuse angle, so its measure must be greater than 90°.
Answer	$m\angle Q = 120°$	

The table below summarizes the special types of quadrilaterals and some of their properties.

Name of Quadrilateral	Quadrilateral	Description
Parallelogram		2 pairs of parallel sides. Opposite sides and opposite angles are congruent.
Rectangle		2 pairs of parallel sides. 4 right angles (90°). Opposite sides are parallel and congruent. All angles are congruent.
Square		4 congruent sides. 4 right angles (90°). Opposite sides are parallel. All angles are congruent.
Trapezoid		Only one pair of opposite sides is parallel.

Summary

A quadrilateral is a mathematical name for a four-sided polygon. Parallelograms, squares, rectangles, and trapezoids are all examples of quadrilaterals. These quadrilaterals earn their distinction based on their properties, including the number of pairs of parallel sides they have and their angle and side measurements.

1.2.1 Self Check Solutions

Self Check A

Which of the following statements is true?

A) Some trapezoids are parallelograms.
B) All trapezoids are quadrilaterals.
C) All rectangles are squares.
D) A shape cannot be a parallelogram and a quadrilateral.

A) Some trapezoids are parallelograms.
Incorrect. Trapezoids have only one pair of parallel sides; parallelograms have two pairs of parallel sides. A trapezoid can never be a parallelogram. The correct answer is that all trapezoids are quadrilaterals.

B) All trapezoids are quadrilaterals.
Correct. Trapezoids are four-sided polygons, so they are all quadrilaterals.

C) All rectangles are squares.
Incorrect. Some rectangles may be squares, but not *all* rectangles have four congruent sides. All squares are rectangles however. The correct answer is that all trapezoids are quadrilaterals.

D) A shape cannot be a parallelogram and a quadrilateral.
Incorrect. All parallelograms are quadrilaterals, so if it is a parallelogram, it is also a quadrilateral. The correct answer is that all trapezoids are quadrilaterals.

1.2.2 Perimeter and Area

Learning Objective(s)
1 Find the perimeter of a polygon.
2 Find the area of a polygon.
3 Find the area and perimeter of non-standard polygons.

Introduction

Perimeter and **area** are two important and fundamental mathematical topics. They help you to quantify physical space and also provide a foundation for more advanced mathematics found in algebra, trigonometry, and calculus. Perimeter is a measurement of the distance around a shape and area gives us an idea of how much surface the shape covers.

Knowledge of area and perimeter is applied practically by people on a daily basis, such as architects, engineers, and graphic designers, and is math that is very much needed by people in general. Understanding how much space you have and learning how to fit shapes together exactly will help you when you paint a room, buy a home, remodel a kitchen, or build a deck.

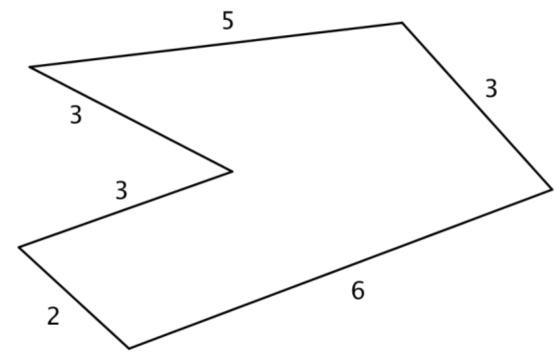
Perimeter

Objective 1

The perimeter of a two-dimensional shape is the distance around the shape. You can think of wrapping a string around a triangle. The length of this string would be the perimeter of the triangle. Or walking around the outside of a park, you walk the distance of the park's perimeter. Some people find it useful to think "peRIMeter" because the edge of an object is its rim and peRIMeter has the word "rim" in it.

If the shape is a **polygon**, then you can add up all the lengths of the sides to find the perimeter. Be careful to make sure that all the lengths are measured in the same units. You measure perimeter in linear units, which is one dimensional. Examples of units of measure for length are inches, centimeters, or feet.

Example	
Problem	**Find the perimeter of the given figure. All measurements indicated are inches.**



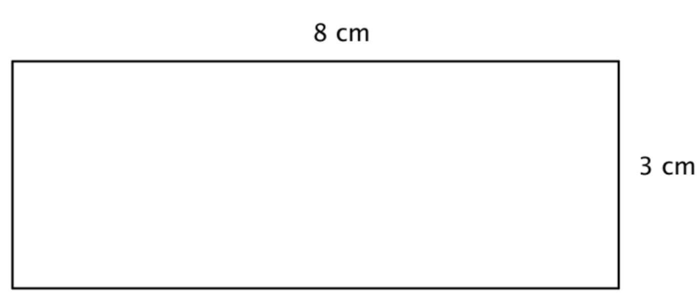
	$P = 5 + 3 + 6 + 2 + 3 + 3$	Since all the sides are measured in inches, just add the lengths of all six sides to get the perimeter.
Answer	P = 22 inches	Remember to include units.

This means that a tightly wrapped string running the entire distance around the polygon would measure 22 inches long.

Example		
Problem	**Find the perimeter of a triangle with sides measuring 6 cm, 8 cm, and 12 cm.**	
	$P = 6 + 8 + 12$	Since all the sides are measured in centimeters, just add the lengths of all three sides to get the perimeter.
Answer	P = 26 centimeters	

Sometimes, you need to use what you know about a polygon in order to find the perimeter. Let's look at the rectangle in the next example.

Example	
Problem	**A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the perimeter.**



	$P = 3 + 3 + 8 + 8$	Since this is a rectangle, the opposite sides have the same lengths, 3 cm. and 8 cm. Add up the lengths of all four sides to find the perimeter.
Answer	$P = 22$ cm	

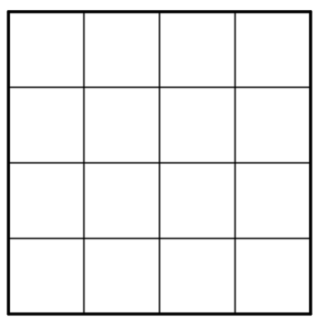
Notice that the perimeter of a rectangle always has two pairs of equal length sides. In the above example you could have also written $P = 2(3) + 2(8) = 6 + 16 = 22$ cm. The formula for the perimeter of a rectangle is often written as $P = 2l + 2w$, where l is the length of the rectangle and w is the width of the rectangle.

Area of Parallelograms

Objective **2**

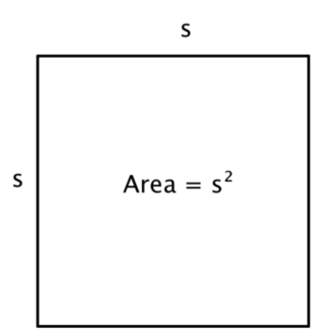
The area of a two-dimensional figure describes the amount of surface the shape covers. You measure area in square units of a fixed size. Examples of square units of measure are square inches, square centimeters, or square miles. When finding the area of a polygon, you count how many squares of a certain size will cover the region inside the polygon.

Let's look at a 4 x 4 square.



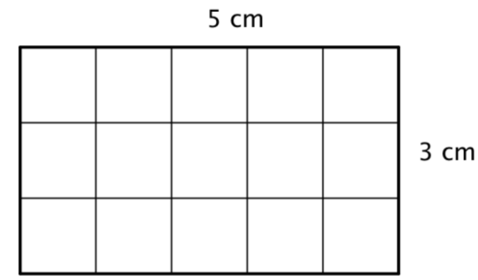
You can count that there are 16 squares, so the area is 16 square units. Counting out 16 squares doesn't take too long, but what about finding the area if this is a larger square or the units are smaller? It could take a long time to count.

Fortunately, you can use multiplication. Since there are 4 rows of 4 squares, you can multiply 4 • 4 to get 16 squares! And this can be generalized to a formula for finding the area of a square with any length, s: *Area* $= s \cdot s = s^2$.

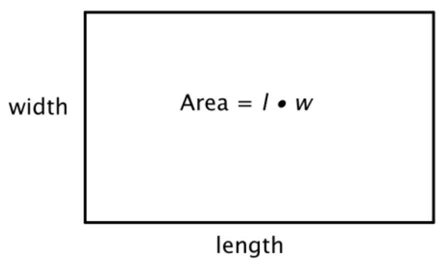


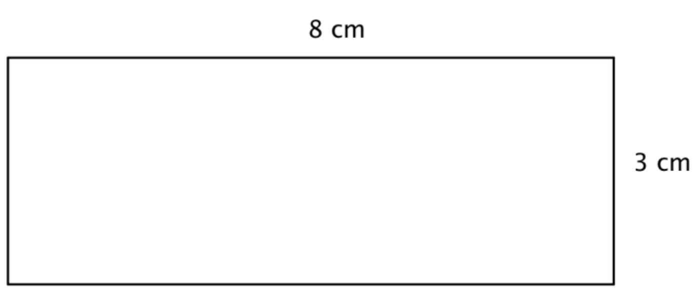
You can write "in^2" for square inches and "ft^2" for square feet.

To help you find the area of the many different categories of polygons, mathematicians have developed formulas. These formulas help you find the measurement more quickly than by simply counting. The formulas you are going to look at are all developed from the understanding that you are counting the number of square units *inside* the polygon. Let's look at a rectangle.



You can count the squares individually, but it is much easier to multiply 3 times 5 to find the number more quickly. And, more generally, the area of any rectangle can be found by multiplying *length* times *width.*

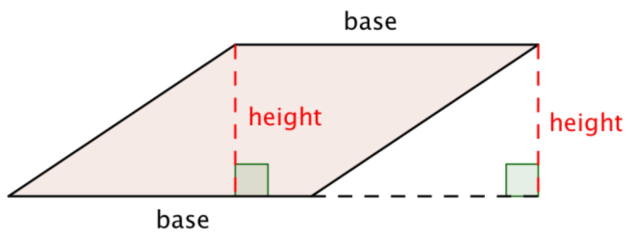


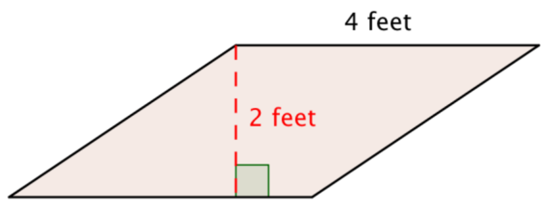
Example		
Problem	A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the area. 	
	$A = l \cdot w$	Start with the formula for the area of a rectangle, which multiplies the length times the width.

	$A = 8 \cdot 3$	Substitute 8 for the length and 3 for the width.
Answer	$A = 24\ cm^2$	Be sure to include the units, in this case square cm.

It would take 24 squares, each measuring 1 cm on a side, to cover this rectangle.

The formula for the area of any parallelogram (remember, a rectangle is a type of parallelogram) is the same as that of a rectangle: *Area* = *l* • *w*. Notice in a rectangle, the length and the width are perpendicular. This should also be true for all parallelograms. *Base* (*b*) for the length (of the base), and *height* (*h*) for the width of the line perpendicular to the base is often used. So the formula for a parallelogram is generally written, *A* = *b* • *h.*



Example		
Problem	**Find the area of the parallelogram.**	
		
	$A = b \cdot h$	Start with the formula for the area of a parallelogram: *Area* = *base* • *height*.
	$A = 4 \cdot 2$	Substitute the values into the formula.
	$A = 8$	Multiply.
Answer	The area of the parallelogram is 8 ft^2.	

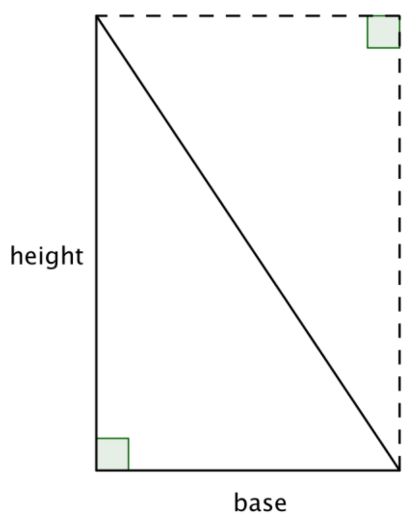
Self Check A
Find the area of a parallelogram with a height of 12 feet and a base of 9 feet.

A) 21 ft^2
B) 54 ft^2
C) 42 ft
D) 108 ft^2

Area of Triangles and Trapezoids

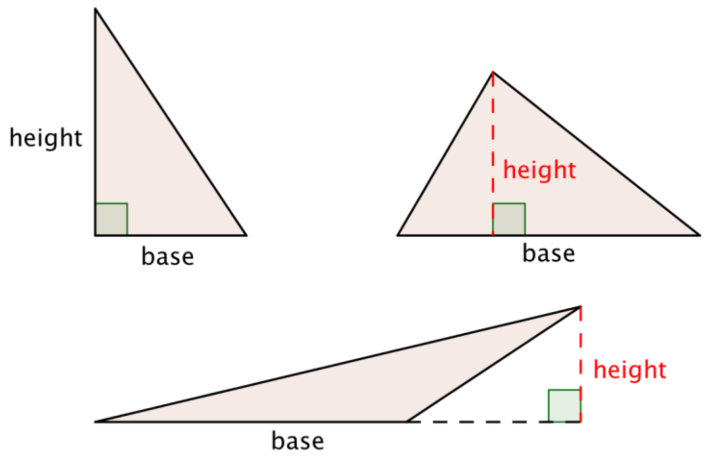
Objective **3**

The formula for the area of a triangle can be explained by looking at a right triangle. Look at the image below—a rectangle with the same height and base as the original triangle. The area of the triangle is one half of the rectangle!



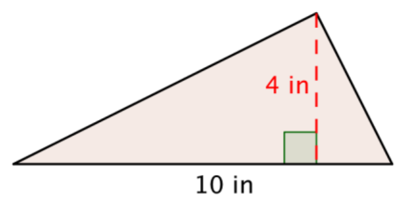
Since the area of two congruent triangles is the same as the area of a rectangle, you can come up with the formula $Area = \frac{1}{2}b \cdot h$ to find the area of a triangle.

When you use the formula for a triangle to find its area, it is important to identify a base and its corresponding height, which is perpendicular to the base.



Example

Problem **A triangle has a height of 4 inches and a base of 10 inches. Find the area.**



$A = \frac{1}{2}bh$ Start with the formula for the area of a triangle.

$A = \frac{1}{2} \bullet 10 \bullet 4$ Substitute 10 for the base and 4 for the height.

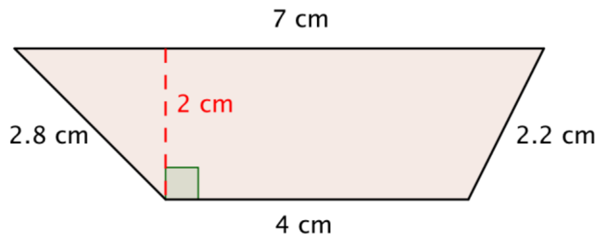
$A = \frac{1}{2} \bullet 40$ Multiply.

$A = 20$

Answer $A = 20 \text{ in}^2$

Now let's look at the trapezoid. To find the area of a trapezoid, take the average length of the two parallel bases and multiply that length by the height: $A = \frac{(b_1 + b_2)}{2}h$.

An example is provided below. Notice that the height of a trapezoid will always be perpendicular to the bases (just like when you find the height of a parallelogram).

Example	
Problem	**Find the area of the trapezoid.**
	
$A = \frac{(b_1 + b_2)}{2}h$	Start with the formula for the area of a trapezoid.
$A = \frac{(4+7)}{2} \bullet 2$ $A = \frac{11}{2} \bullet 2$ $A = 11$	Substitute 4 and 7 for the bases and 2 for the height, and find *A*.
Answer	The area of the trapezoid is 11 cm^2.

Area Formulas

Use the following formulas to find the areas of different shapes.

square: $A = s^2$

rectangle: $A = l \bullet w$

parallelogram: $A = b \bullet h$

triangle: $A = \frac{1}{2} b \bullet h$

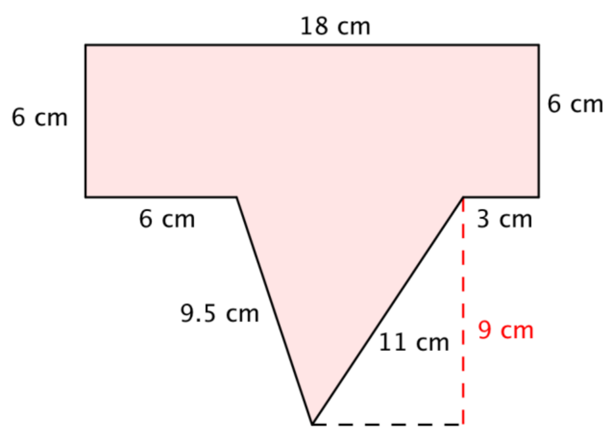
trapezoid: $A = \frac{(b_1 + b_2)}{2}h$

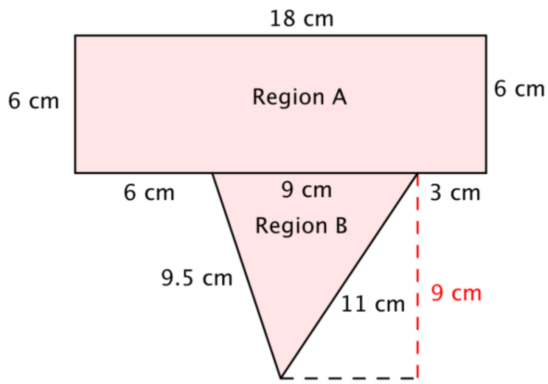
Working with Perimeter and Area

Often you need to find the area or perimeter of a shape that is not a standard polygon. Artists and architects, for example, usually deal with complex shapes. However, even complex shapes can be thought of as being composed of smaller, less complicated shapes, like rectangles, trapezoids, and triangles.

To find the perimeter of non-standard shapes, you still find the distance around the shape by adding together the length of each side.

Finding the area of non-standard shapes is a bit different. You need to create regions *within* the shape for which you can find the area, and add these areas together. Have a look at how this is done below.

Example		
Problem	**Find the area and perimeter of the polygon.** 	
	$P = 18 + 6 + 3 + 11 + 9.5 + 6 + 6$ $P = 59.5$ cm	To find the perimeter, add together the lengths of the sides. Start at the top and work clockwise around the shape.



To find the area, divide the polygon into two separate, simpler regions. The area of the entire polygon will equal the sum of the areas of the two regions.

Area of Polygon = (*Area of A*) + (*Area of B*)

$$\text{Area of Region } A = l \bullet w$$
$$= 18 \bullet 6$$
$$= 108$$

Region A is a rectangle. To find the area, multiply the length (18) by the width (6).

The area of Region A is 108 cm^2.

$$\text{Area of Region B} = \frac{1}{2} b \bullet h$$
$$= \frac{1}{2} \bullet 9 \bullet 9$$
$$= \frac{1}{2} \bullet 81$$
$$= 40.5$$

Region B is a triangle. To find the area, use the formula $\frac{1}{2}bh$, where the base is 9 and the height is 9.

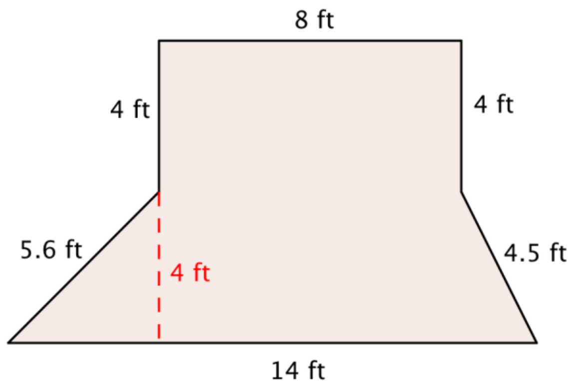
The area of Region B is 40.5 cm^2.

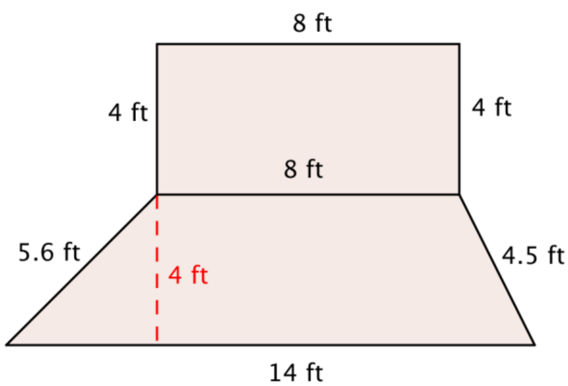
108 cm^2 + 40.5 cm^2 = 148.5 cm^2. Add the regions together.

Answer *Perimeter* = 59.5 cm
Area = 148.5 cm^2

You also can use what you know about perimeter and area to help solve problems about situations like buying fencing or paint, or determining how big a rug is needed in the living room. Here’s a fencing example.

Example	
Problem	**Rosie is planting a garden with the dimensions shown below. She wants to put a thin, even layer of mulch over the entire surface of the garden. The mulch costs $3 a square foot. How much money will she have to spend on mulch?**





This shape is a combination of two simpler shapes: a rectangle and a trapezoid. Find the area of each.

$$A = l \bullet w$$
$$A = 8 \bullet 4$$
$$A = 32\ ft^2$$

Find the area of the rectangle.

$$A = \frac{(b_1 + b_2)}{2}h$$
$$A = \frac{(14 + 8)}{2} \bullet 4$$
$$A = \frac{22}{2} \bullet 4$$
$$A = 11 \bullet 4$$
$$A = 44\ ft^2$$

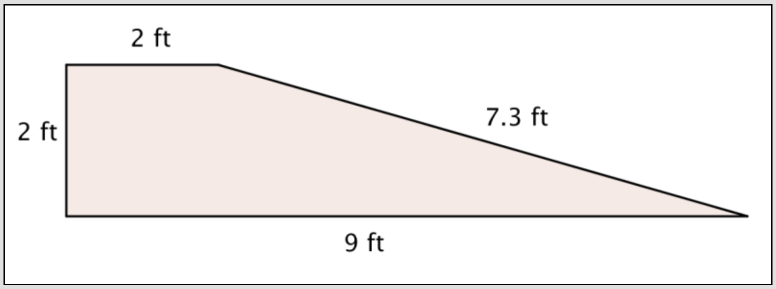
Find the area of the trapezoid.

$32\ ft^2 + 44\ ft^2 = 76\ ft^2$ Add the measurements.

$76\ ft^2 \bullet \$3 = \$228$ Multiply by \$3 to find out how much Rosie will have to spend.

Answer Rosie will spend \$228 to cover her garden with mulch.

Self Check B
Find the area of the shape shown below.



A) 11 ft^2
B) 18 ft^2
C) 20.3 ft
D) 262.8 ft^2

Summary

The perimeter of a two-dimensional shape is the distance around the shape. It is found by adding up all the sides (as long as they are all the same unit). The area of a two-dimensional shape is found by counting the number of squares that cover the shape. Many formulas have been developed to quickly find the area of standard polygons, like triangles and parallelograms.

1.2.2 Self Check Solutions

Self Check A
Find the area of a parallelogram with a height of 12 feet and a base of 9 feet.

A) 21 ft^2
B) 54 ft^2
C) 42 ft
D) 108 ft^2

A) 21 ft^2
Incorrect. It looks like you added the dimensions; remember that to find the area, you multiply the base by the height. The correct answer is 108 ft^2.

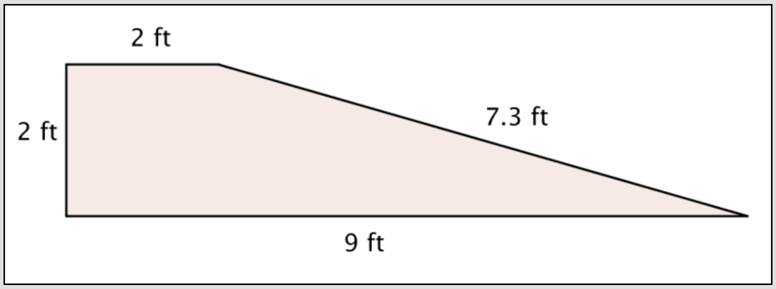
B) 54 ft^2
Incorrect. It looks like you multiplied the base by the height and then divided by 2. To find the area of a parallelogram, you multiply the base by the height. The correct answer is 108 ft^2.

C) 42 ft
Incorrect. It looks like you added 12 + 12 + 9 + 9. This would give you the perimeter of a 12 by 9 rectangle. To find the area of a parallelogram, you multiply the base by the height. The correct answer is 108 ft^2.

D) 108 ft^2
Correct. The height of the parallelogram is 12 and the base of the parallelogram is 9; the area is 12 times 9, or 108 ft^2.

Self Check B

Find the area of the shape shown below.



A) 11 ft^2
B) 18 ft^2
C) 20.3 ft
D) 262.8 ft^2

A) 11 ft^2

Correct. This shape is a trapezoid, so you can use the formula $A = \frac{(b_1 + b_2)}{2}h$ to find the area: $A = \frac{(2+9)}{2} \bullet 2$.

B) 18 ft^2
Incorrect. It looks like you multiplied 2 by 9 to get 18 ft^2; this would work if the shape was a rectangle. This shape is a trapezoid, though, so use the formula $A = \frac{(b_1 + b_2)}{2}h$. The correct answer is 11 ft^2.

C) 20.3 ft
Incorrect. It looks like you added all the dimensions together. This would give you the perimeter. To find the area of a trapezoid, use the formula $A = \frac{(b_1 + b_2)}{2}h$. The correct answer is 11 ft^2.

D) 262.8 ft^2
Incorrect. It looks like you multiplied all of the dimensions together. This shape is a trapezoid, so you use the formula $A = \frac{(b_1 + b_2)}{2}h$. The correct answer is 11 ft^2.

1.2.3 Circles

Learning Objective(s)
1 Identify properties of circles.
2 Find the circumference of a circle.
3 Find the area of a circle.
4 Find the area and perimeter of composite geometric figures.

Introduction

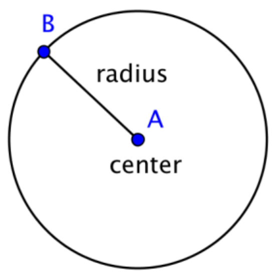
Circles are a common shape. You see them all over—wheels on a car, Frisbees passing through the air, compact discs delivering data. These are all circles.

A circle is a two-dimensional figure just like polygons and quadrilaterals. However, circles are measured differently than these other shapes—you even have to use some different terms to describe them. Let's take a look at this interesting shape.

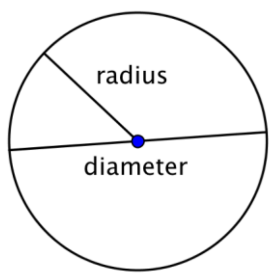
Objective 1

Properties of Circles

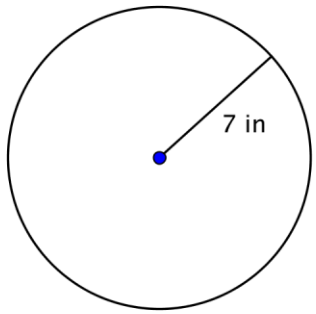
A circle represents a set of points, all of which are the same distance away from a fixed, middle point. This fixed point is called the center. The distance from the center of the circle to any point on the circle is called the **radius**.

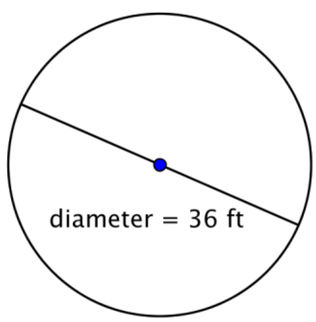


When two radii (the plural of radius) are put together to form a line segment across the circle, you have a **diameter**. The diameter of a circle passes through the center of the circle and has its endpoints on the circle itself.



The diameter of any circle is two times the length of that circle's radius. It can be represented by the expression 2*r*, or "two times the radius." So if you know a circle's radius, you can multiply it by 2 to find the diameter; this also means that if you know a circle's diameter, you can divide by 2 to find the radius.

Example		
Problem	**Find the diameter of the circle.**	
		
	$d = 2r$ $d = 2(7)$ $d = 14$	The diameter is two times the radius, or 2*r*. The radius of this circle is 7 inches, so the diameter is 2(7) = 14 inches.
Answer	The diameter is 14 inches.	

Example	
Problem	**Find the radius of the circle.**
	

	$r = \frac{1}{2}d$ $r = \frac{1}{2}(36)$ $r = 18$	The radius is half the diameter, or $\frac{1}{2}d$. The diameter of this circle is 36 feet, so the radius is $\frac{1}{2}(36) = 18$ feet.
Answer	The radius is 18 feet.	

Circumference

Objective **2**

The distance around a circle is called the **circumference**. (Recall, the distance around a polygon is the perimeter.)

One interesting property about circles is that the ratio of a circle's circumference and its diameter is the same for all circles. No matter the size of the circle, the ratio of the circumference and diameter will be the same.

Some actual measurements of different items are provided below. The measurements are accurate to the nearest millimeter or quarter inch (depending on the unit of measurement used). Look at the ratio of the circumference to the diameter for each one—although the items are different, the ratio for each is approximately the same.

Item	**Circumference (*C*) (rounded to nearest hundredth)**	**Diameter (*d*)**	**Ratio $\frac{C}{d}$**
Cup	253 mm	79 mm	$\frac{253}{79} = 3.2025...$
Quarter	84 mm	27 mm	$\frac{84}{27} = 3.1111...$
Bowl	37.25 in	11.75 in	$\frac{37.25}{11.75} = 3.1702...$

The circumference and the diameter are approximate measurements, since there is no precise way to measure these dimensions exactly. If you were able to measure them more precisely, however, you would find that the ratio $\frac{C}{d}$ would move towards 3.14 for each of the items given. The mathematical name for the ratio $\frac{C}{d}$ is **pi**, and is represented by the Greek letter π.

π is a non-terminating, non-repeating decimal, so it is impossible to write it out completely. The first 10 digits of π are 3.141592653; it is often rounded to 3.14 or estimated as the fraction $\frac{22}{7}$. Note that both 3.14 and $\frac{22}{7}$ are *approximations* of π, and are used in calculations where it is not important to be precise.

Since you know that the ratio of circumference to diameter (or π) is consistent for all circles, you can use this number to find the circumference of a circle if you know its diameter.

$$\frac{C}{d} = \pi \text{, so } C = \pi d$$

Also, since $d = 2r$, then $C = \pi d = \pi(2r) = 2\pi r$.

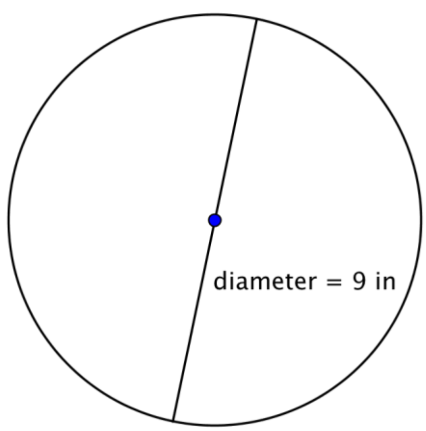
Circumference of a Circle

To find the circumference (C) of a circle, use one of the following formulas:

If you know the diameter (d) of a circle: $C = \pi d$
If you know the radius (r) of a circle: $C = 2\pi r$

Example

Problem **Find the circumference of the circle.**



$C = \pi d$
$C = \pi \bullet 9$
$C \approx 3.14 \bullet 9$
$C \approx 28.26$

To calculate the circumference given a diameter of 9 inches, use the formula $C = \pi d$. Use 3.14 as an approximation for π.

Since you are using an approximation for π, you cannot give an exact measurement of the circumference. Instead, you use the symbol $\approx$ to indicate "approximately equal to."

Answer The circumference is 9π or approximately 28.26 inches.

Example		
Problem	**Find the circumference of a circle with a radius of 2.5 yards.**	
	$C = 2\pi r$ $C = 2\pi \cdot 2.5$ $C = \pi \cdot 5$ $C \approx 3.14 \cdot 5$ $C \approx 15.7$	To calculate the circumference of a circle given a radius of 2.5 yards, use the formula $C = 2\pi r$. Use 3.14 as an approximation for π .
Answer	The circumference is 5π or approximately 15.7 yards.	

Self Check A

A circle has a radius of 8 inches. What is its circumference, rounded to the nearest inch?

A) 25 inches
B) 50 inches
C) 64 inches^2
D) 201 inches

Objective **3**

Area

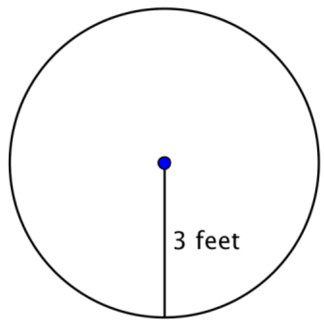
π is an important number in geometry. You have already used it to calculate the circumference of a circle. You use π when you are figuring out the *area* of a circle, too.

Area of a Circle

To find the area (A) of a circle, use the formula: $A = \pi r^2$

Example

Problem **Find the area of the circle.**



$A = \pi r^2$

$A = \pi \bullet 3^2$

$A = \pi \bullet 9$

$A \approx 3.14 \bullet 9$

$A \approx 28.26$

To find the area of this circle, use the formula $A = \pi r^2$.

Remember to write the answer in terms of square units, since you are finding the area.

Answer The area is 9π or approximately 28.26 feet^2.

Self Check B

A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of π.

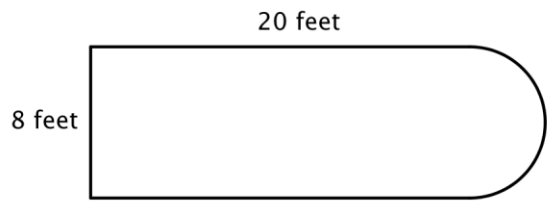
A) 62.8 mm
B) 314 mm^2
C) 400 mm^2
D) 1256 mm^2

Composite Figures

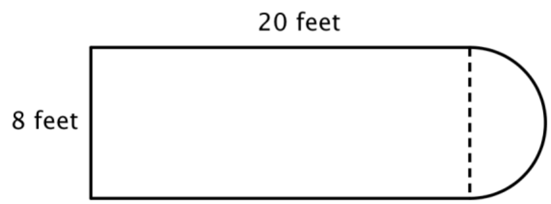
Objective 4

Now that you know how to calculate the circumference and area of a circle, you can use this knowledge to find the perimeter and area of composite figures. The trick to figuring out these types of problems is to identify shapes (and parts of shapes) within the composite figure, calculate their individual dimensions, and then add them together.

For example, look at the image below. Is it possible to find the perimeter?

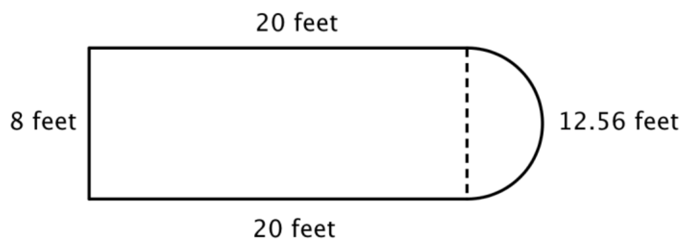


The first step is to identify simpler figures within this composite figure. You can break it down into a rectangle and a semicircle, as shown below.

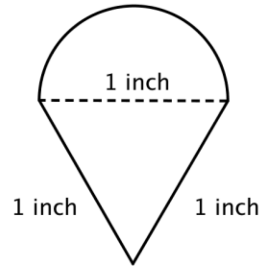


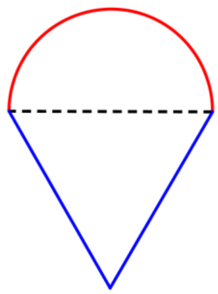
You know how to find the perimeter of a rectangle, and you know how to find the circumference of a circle. Here, the perimeter of the three solid sides of the rectangle is 8 + 20 + 20 = 48 feet. (Note that only three sides of the rectangle will add into the perimeter of the composite figure because the other side is not at an edge; it is covered by the semicircle!)

To find the circumference of the semicircle, use the formula $C = \pi d$ with a diameter of 8 feet, then take half of the result. The circumference of the semicircle is 4π, or approximately 12.56 feet, so the total perimeter is about 60.56 feet.



Example	
Problem	**Find the perimeter (to the nearest hundredth) of the composite figure, made up of a semi-circle and a triangle.**





Identify smaller shapes within the composite figure. This figure contains a semicircle and a triangle.

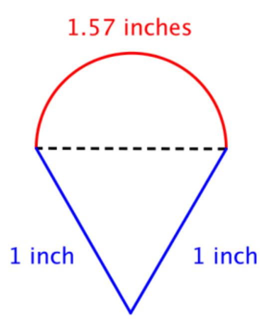
Diameter (d) = 1

$$C = \pi d$$
$$C = \pi(1)$$
$$C = \pi$$

Circumference of semicircle = $\frac{1}{2}\pi$ or approximately 1.57 inches

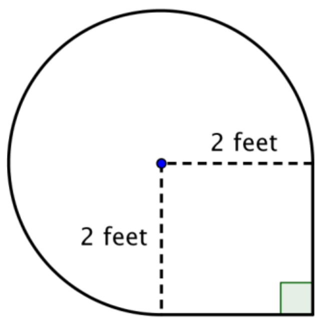
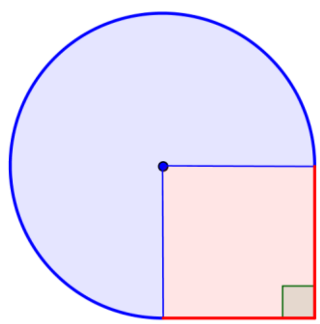
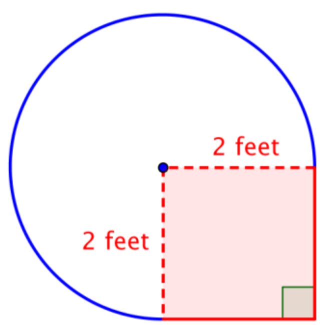
Find the circumference of the circle. Then divide by 2 to find the circumference of the semi-circle.

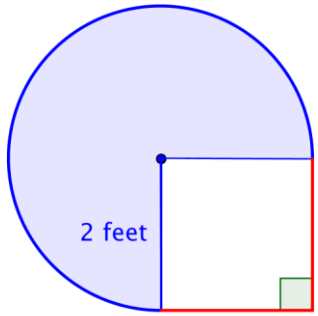
$1 + 1 + \frac{1}{2}\pi \approx 3.57$ inches



Find the total perimeter by adding the circumference of the semicircle and the lengths of the two legs. Since our measurement of the semi-circle's circumference is approximate, the perimeter will be an approximation also.

Answer Approximately 3.57 inches

Example	
Problem	**Find the area of the composite figure, made up of three-quarters of a circle and a square, to the nearest hundredth.**
	
	Identify smaller shapes within the composite figure. This figure contains a circular region and a square. If you find the area of each, you can find the area of the entire figure.
	Find the area of the square.
$\text{Area of square} = s^2$ $= (2)^2$ $= 4\ ft^2$	



Find the area of the circular region. The radius is 2 feet.

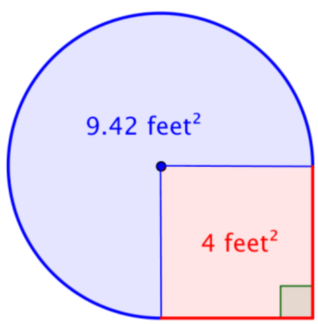
Note that the region is $\frac{3}{4}$ of a whole circle, so you need to multiply the area of the circle by $\frac{3}{4}$.

Use 3.14 as an approximation for π.

$$\begin{aligned}\text{Area of full circle} &= \pi r^2 \\ &= \pi(2)^2 \\ &= 4\pi\ ft^2\end{aligned}$$

$$\begin{aligned}\text{Area of region} &= \frac{3}{4} \bullet 4\pi \\ &= 3\pi \\ &\approx 3 \bullet 3.14\ ft^2\end{aligned}$$

This is approximately 9.42 feet^2.



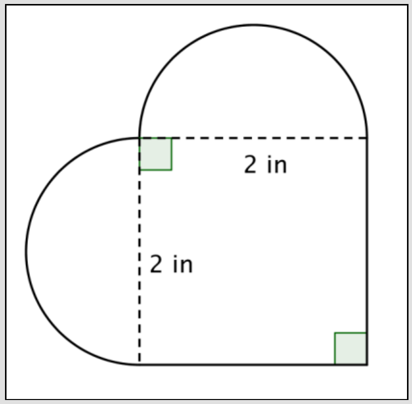
Add the two regions together. Since your measurement of the circular's area is approximate, the area of the figure will be an approximation also.

4 feet^2 + 3π feet^2 = approximately 13.42 feet^2

Answer The area is approximately 13.42 feet^2.

Self Check C

What is the area (to the nearest hundredth) of the figure shown below? (Both rounded regions are semi-circles.)



A) 16.56 in^2
B) 7.14 in^2
C) 4 in^2
D) 3.14 in^2

Summary

Circles are an important geometric shape. The distance around a circle is called the circumference, and the interior space of a circle is called the area. Calculating the circumference and area of a circle requires a number called pi (π), which is a non-terminating, non-repeating decimal. Pi is often approximated by the values 3.14 and $\frac{22}{7}$. You can find the perimeter or area of composite shapes—including shapes that contain circular sections—by applying the circumference and area formulas where appropriate.

1.2.3 Self Check Solutions

Self Check A

A circle has a radius of 8 inches. What is its circumference, rounded to the nearest inch?

A) 25 inches
B) 50 inches
C) 64 $inches^2$
D) 201 inches

A) 25 inches
Incorrect. You multiplied the radius times π; the correct formula for circumference when the radius is given is $C = 2\pi r$ The correct answer is 50 inches.

B) 50 inches
Correct. If the radius is 8 inches, the correct formula for circumference when the radius

is given is $C = 2\pi r$ The correct answer is 50 inches.

C) 64 inches^2
Incorrect. You squared 8 inches to arrive at the answer 64 inches^2; this will give you the area of a square with sides of 8 inches. Remember that the formula for circumference when the radius is given is $C = 2\pi r$. The correct answer is 50 inches.

D) 201 inches
Incorrect. It looks like you squared 8 and then multiplied 64 by π to arrive at this answer. Remember that the formula for circumference when the radius is given is $C = 2\pi r$. The correct answer is 50 inches.

Self Check B
A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of π .

A) 62.8 mm
B) 314 mm^2
C) 400 mm^2
D) 1256 mm^2

A) 62.8 mm
Incorrect. You found the circumference of the button: 20 • 3.14 = 62.8. To find the area, use the formula $A = \pi r^2$. The correct answer is 314 mm^2.

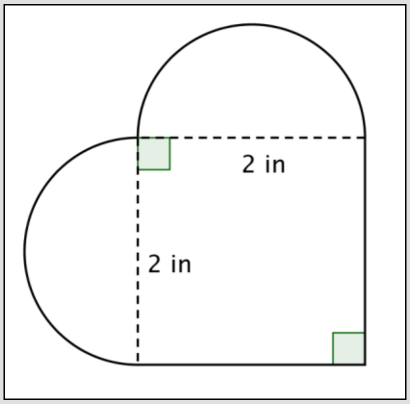
B) 314 mm^2
Correct. The diameter is 20 mm, so the radius must be 10 mm. Then, using the formula $A = \pi r^2$, you find $A = \pi \bullet 10^2 = \pi \bullet 100 \approx 314$ mm^2.

C) 400 mm^2
Incorrect. You squared 20 to arrive at 400 mm^2; this gives you the area of a square with sides of length 20, not the area of a circle. To find the area, use the formula $A = \pi r^2$. The correct answer is 314 mm^2.

D) 1256 mm^2
Incorrect. It looks like you squared 20 and then multiplied by π . 20 is the diameter, not the radius! To find the area, use the formula $A = \pi r^2$. The correct answer is 314 mm^2.

Self Check C

What is the area (to the nearest hundredth) of the figure shown below? (Both rounded regions are semi-circles.)



A) 16.56 in^2
B) 7.14 in^2
C) 4 in^2
D) 3.14 in^2

A) 16.56 in^2
Incorrect. It looks like you calculated the area of a circle using a radius of 2; in this figure, the radius of each circle is 1. To find the area of the figure, imagine the two semi-circles are put together to create one circle. Then calculate the area of the circle and add it to the area of the square. The correct answer is 7.14 in^2.

B) 7.14 in^2
Correct. Imagine the two semi-circles being put together to create one circle. The radius of the circle is 1 inch; this means the area of the circle is $\pi r^2 = \pi \bullet 1^2 = \pi$. The area of the square is 2 • 2 = 4. Adding those together yields 7.14 in^2.

C) 4 in^2
Incorrect. It looks like you calculated the area of the square, but not the circle. Imagine the two semi-circles are put together to create one circle. Then calculate the area of the circle and add it to the area of the square. The correct answer is 7.14 in^2.

D) 3.14 in^2
Incorrect. It looks like you calculated the area of the circle, but not the square. Calculate the area of the square and add it to the area of the circle. The correct answer is 7.14 in^2.

1.3.1 Solids

Learning Objective(s)
1 Identify geometric solids.
2 Find the volume of geometric solids.
3 Find the volume of a composite geometric solid.

Introduction

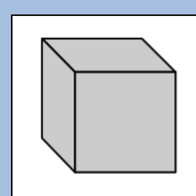
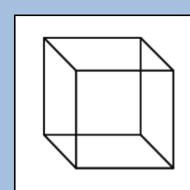
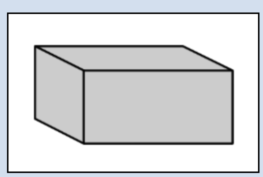
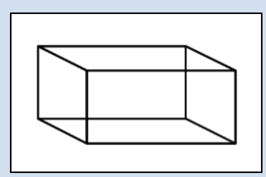
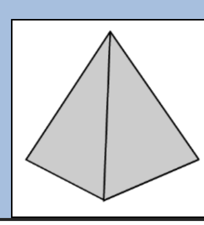
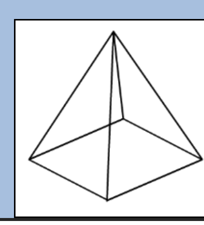
Living in a two-dimensional world would be pretty boring. Thankfully, all of the physical objects that you see and use every day—computers, phones, cars, shoes—exist in three dimensions. They all have length, width, and height. (Even very thin objects like a piece of paper are three-dimensional. The thickness of a piece of paper may be a fraction of a millimeter, but it does exist.)

In the world of geometry, it is common to see three-dimensional figures. In mathematics, a flat side of a three-dimensional figure is called a **face**. **Polyhedrons** are shapes that have four or more faces, each one being a polygon. These include cubes, prisms, and pyramids. Sometimes you may even see single figures that are composites of two of these figures. Let's take a look at some common polyhedrons.

Identifying Solids

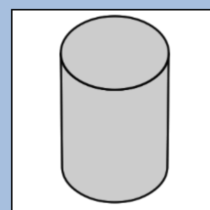
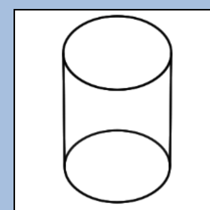
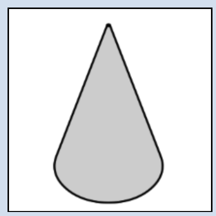
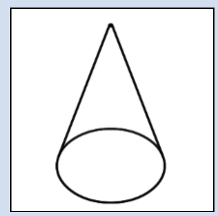
Objective 1

The first set of solids contains rectangular bases. Have a look at the table below, which shows each figure in both solid and transparent form.

Name	Definition	Solid Form	Transparent Form
Cube	A six-sided polyhedron that has congruent squares as faces.		
Rectangular prism	A polyhedron that has three pairs of congruent, rectangular, parallel faces.		
Pyramid	A polyhedron with a polygonal base and a collection of triangular faces that meet at a point.		

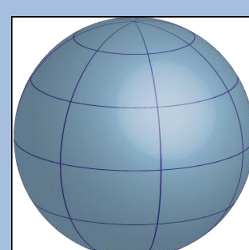
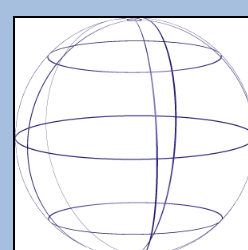
Notice the different names that are used for these figures. A **cube** is different than a square, although they are sometimes confused with each other—a cube has three dimensions, while a square only has two. Likewise, you would describe a shoebox as a **rectangular prism** (not simply a rectangle), and the ancient **pyramids** of Egypt as…well, as pyramids (not triangles)!

In this next set of solids, each figure has a circular base.

Name	Definition	Solid Form	Transparent Form
Cylinder	A solid figure with a pair of circular, parallel bases and a round, smooth face between them.		
Cone	A solid figure with a single circular base and a round, smooth face that diminishes to a single point.		

Take a moment to compare a pyramid and a **cone**. Notice that a pyramid has a rectangular base and flat, triangular faces; a cone has a circular base and a smooth, rounded body.

Finally, let's look at a shape that is unique: a **sphere**.

Name	Definition	Solid Form	Transparent Form
Sphere	A solid, round figure where every point on the surface is the same distance from the center.		

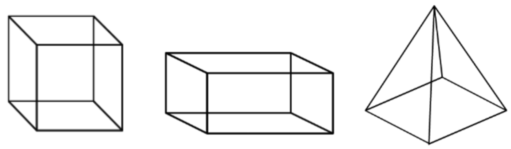
There are many spherical objects all around you—soccer balls, tennis balls, and baseballs being three common items. While they may not be perfectly spherical, they are generally referred to as spheres.

Example

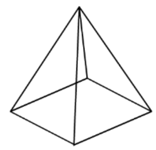
Problem **A three-dimensional figure has the following properties:**

- **It has a rectangular base.**
- **It has four triangular faces.**

What kind of a solid is it?



A rectangular base indicates that it must be a cube, rectangular prism, or pyramid.



Since the faces are triangular, it must be a pyramid.

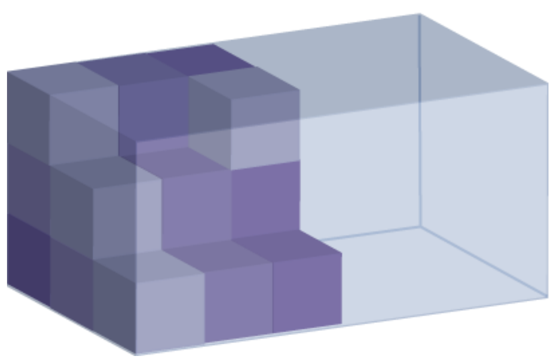
Answer The solid is a pyramid.

Objective **2**

Volume

Recall that perimeter measures one dimension (length), and area measures two dimensions (length and width). To measure the amount of space a three-dimensional figure takes up, you use another measurement called **volume**.

To visualize what "volume" measures, look back at the transparent image of the rectangular prism mentioned earlier (or just think of an empty shoebox). Imagine stacking identical cubes inside that box so that there are no gaps between any of the cubes. Imagine filling up the entire box in this manner. If you counted the number of cubes that fit inside that rectangular prism, you would have its volume.

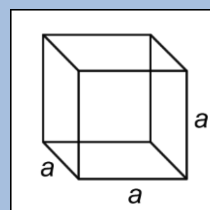
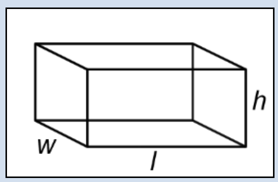
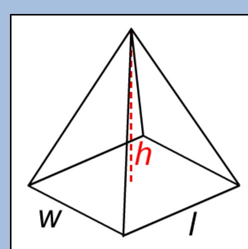


Volume is measured in cubic units. The shoebox illustrated above may be measured in cubic inches (usually represented as in^3 or $inches^3$), while the Great Pyramid of Egypt would be more appropriately measured in cubic meters (m^3 or $meters^3$).

To find the volume of a geometric solid, you could create a transparent version of the solid, create a bunch of 1x1x1 cubes, and then stack them carefully inside. However, that would take a long time! A much easier way to find the volume is to become familiar with some geometric formulas, and to use those instead.

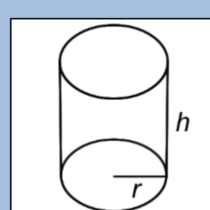
Let's go through the geometric solids once more and list the volume formula for each.

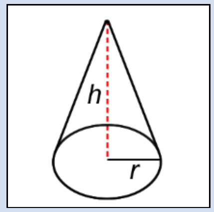
As you look through the list below, you may notice that some of the volume formulas look similar to their area formulas. To find the volume of a rectangular prism, you find the area of the base and then multiply that by the height.

Name	Transparent Form	Volume Formula
Cube		$V = a \bullet a \bullet a = a^3$ a = the length of one side
Rectangular prism		$V = l \bullet w \bullet h$ l = length w = width h = height
Pyramid		$V = \frac{l \bullet w \bullet h}{3}$ l = length w = width h = height

Remember that all cubes are rectangular prisms, so the formula for finding the volume of a cube is the area of the base of the cube times the height.

Now let's look at solids that have a circular base.

Name	Transparent Form	Volume Formula
Cylinder		$V = \pi \bullet r^2 \bullet h$ r = radius h = height

Cone		$V = \dfrac{\pi \bullet r^2 \bullet h}{3}$ r = radius h = height

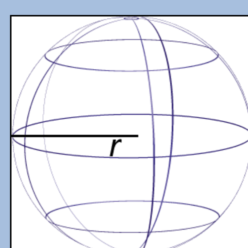
Here you see the number π again.

The volume of a **cylinder** is the area of its base, πr^2, times its height, h.

Compare the formula for the volume of a cone ($V = \dfrac{\pi \bullet r^2 \bullet h}{3}$) with the formula for the volume of a pyramid ($V = \dfrac{l \bullet w \bullet h}{3}$). The numerator of the cone formula is the volume formula for a cylinder, and the numerator of the pyramid formula is the volume formula for a rectangular prism. Then divide each by 3 to find the volume of the cone and the pyramid. Looking for patterns and similarities in the formulas can help you remember which formula refers to a given solid.

Finally, the formula for a sphere is provided below. Notice that the radius is cubed, not squared and that the quantity πr^3 is multiplied by

$\dfrac{4}{3}$.

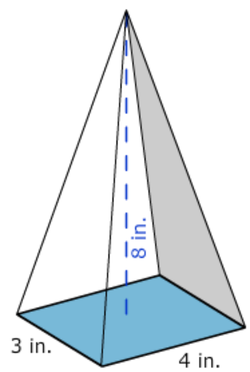
Name	Wireframe Form	Volume Formula
Sphere		$V = \dfrac{4}{3}\pi r^3$ r = radius

Applying the Formulas

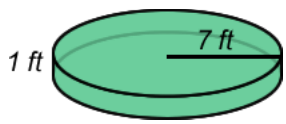
You know how to identify the solids, and you also know the volume formulas for these solids. To calculate the actual volume of a given shape, all you need to do is substitute the solid’s dimensions into the formula and calculate.

In the examples below, notice that cubic units (meters^3, inches^3, feet^3) are used.

Example		
Problem	**Find the volume of a cube with side lengths of 6 meters.**	
	$V = a \cdot a \cdot a = a^3$	Identify the proper formula to use.
	a = side length	
	$V = 6 \cdot 6 \cdot 6 = 6^3$	Substitute a = 6 into the formula.
	$6 \cdot 6 \cdot 6 = 216$	Calculate the volume.
Answer	Volume = 216 meters^3	

Example		
Problem	**Find the volume of the shape shown below.**	
		
	Pyramid.	Identify the shape. It has a rectangular base and rises to a point, so it is a pyramid.
	$V = \frac{l \cdot w \cdot h}{3}$	Identify the proper formula to use.
	l = length w = width h = height	
	4 = length 3 = width 8 = height $V = \frac{4 \cdot 3 \cdot 8}{3}$	Use the image to identify the dimensions. Then substitute l = 4, w = 3, and h = 8 into the formula.

	$V = \dfrac{96}{3}$ $= 32$	Calculate the volume.
Answer	The volume of the pyramid is 32 inches^3.	

Example		
Problem	**Find the volume of the shape shown below.** **Use 3.14 for π, and round the answer to the nearest hundredth.** 	
	Cylinder.	Identify the shape. It has a circular base and has uniform thickness (or height), so it is a cylinder.
	$V = \pi \bullet r^2 \bullet h$	Identify the proper formula to use.
	$V = \pi \bullet 7^2 \bullet 1$	Use the image to identify the dimensions. Then substitute r = 7 and h = 1 into the formula.
	$V = \pi \bullet 49 \bullet 1$ $= 49\pi$ ≈ 153.86	Calculate the volume, using 3.14 as an approximation for π.
Answer	The volume is 49π or approximately 153.86 feet^3.	

Self Check A

Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

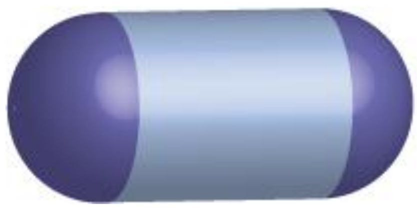
A) 24 inches^2
B) 30 inches^2
C) 240 inches^3
D) 720 inches^3

Composite Solids

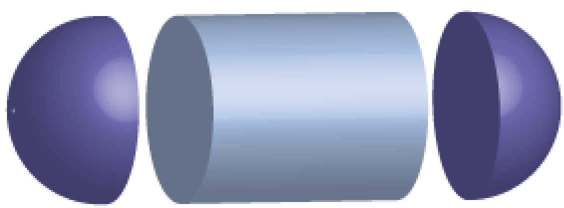
Objective **3**

Composite geometric solids are made from two or more geometric solids. You can find the volume of these solids as well, as long as you are able to figure out the individual solids that make up the composite shape.

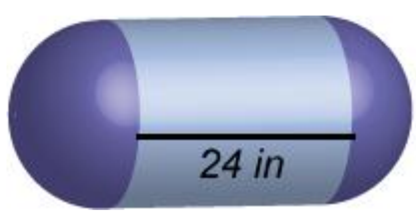
Look at the image of a capsule below. Each end is a half-sphere. You can find the volume of the solid by taking it apart. What solids can you break this shape into?

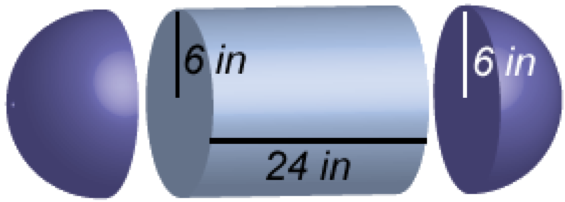


You can break it into a cylinder and two half-spheres.



Two half-spheres form a whole one, so if you know the volume formulas for a cylinder and a sphere, you can find the volume of this capsule.

Example	
Problem	**If the radius of the spherical ends is 6 inches, find the volume of the solid below. Use 3.14 for π. Round your final answer to the nearest whole number.** 



Identify the composite solids. This capsule can be thought of as a cylinder with a half-sphere on each end.

Volume of a cylinder: $\pi \bullet r^2 \bullet h$

Volume of a sphere: $\frac{4}{3}\pi r^3$

Identify the proper formulas to use.

Volume of a cylinder: $\pi \bullet 6^2 \bullet 24$

Volume of a sphere: $\frac{4}{3}\pi \bullet 6^3$

Substitute the dimensions into the formulas.

The height of a cylinder refers to the section between the two circular bases. This dimension is given as 24 inches, so h = 24.

The radius of the sphere is 6 inches. You can use r = 6 in both formulas.

Volume of the cylinder:
$$\begin{aligned} V &= \pi \bullet 36 \bullet 24 \\ &= 864 \bullet \pi \\ &\approx 2712.96 \end{aligned}$$

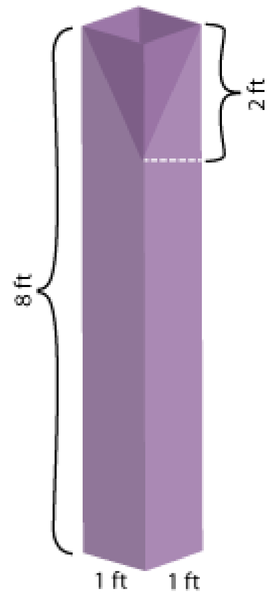
Calculate the volume of the cylinder and the sphere.

Volume of the sphere:
$$\begin{aligned} V &= \frac{4}{3}\pi \bullet 216 \\ &= 288 \bullet \pi \\ &\approx 904.32 \end{aligned}$$

Volume of capsule: Add the volumes.

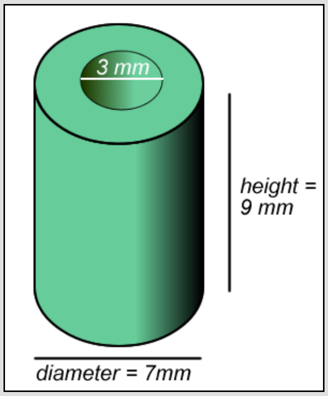
2712.96 + 904.32 $\approx$ 3617.28

Answer The volume of the capsule is 1,152π or approximately 3617 inches^3.

Example		
Problem	**A sculptor carves a rectangular prism out of a solid piece of wood. Then, at the top, she hollows out an inverted pyramid. The solid, and its dimensions, are shown at right. What is the volume of the finished piece?**	
		Identify the composite solids. This sculpture can be thought of as a rectangular prism with a pyramid removed.
	Volume of rectangular prism: $l \cdot w \cdot h$ Volume of pyramid: $\dfrac{l \cdot w \cdot h}{3}$	Identify the proper formulas to use.
	Volume of rectangular prism: $1 \cdot 1 \cdot 8 = 8$ Volume of pyramid: $\dfrac{1 \cdot 1 \cdot 2}{3} = \dfrac{2}{3}$	Substitute the dimensions into the formulas, and calculate.
	Volume of sculpture: $V = 8 - \dfrac{2}{3}$ $= 7\dfrac{1}{3}$	Subtract the volume of the pyramid from the volume of the rectangular prism.
Answer	The volume of the sculpture is $7\frac{1}{3}$ feet^3.	

Self Check B

A machine takes a solid cylinder with a height of 9 mm and a diameter of 7 mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3 mm. Which of the following expressions would correctly find the volume of the solid?



A) $(\pi \bullet 7^2 \bullet 9) - (\pi \bullet 3^2 \bullet 9)$

B) $(\pi \bullet 3.5^2 \bullet 9) - (\pi \bullet 1.5^2 \bullet 9)$

C) $(\pi \bullet 7^2 \bullet 9) + (\pi \bullet 3^2 \bullet 9)$

D) $(\pi \bullet 3.5^2 \bullet 9) + (\pi \bullet 1.5^2 \bullet 9)$

Summary

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

1.3.1 Self Check Solutions

Self Check A

Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

A) 24 inches^2
B) 30 inches^2
C) 240 inches^3
D) 720 inches^3

A) 24 inches^2
Incorrect. It looks like you multiplied 8 inches by 3 inches to arrive at 24 inches^2. This is the area of the base, not the volume of the rectangular prism. The correct answer is 240 inches^3.

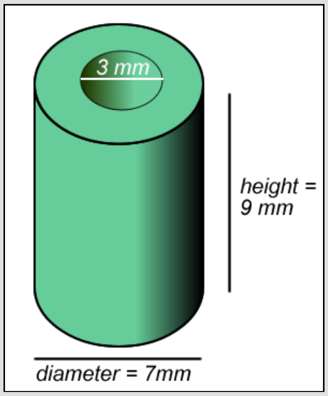
B) 30 inches^2
Incorrect. It looks like you multiplied 3 inches by 10 inches to arrive at 30 inches^2. This is the area of one face of the rectangular prism, not the volume of the rectangular prism. The correct answer is 240 inches^3.

C) 240 inches^3
Correct. To find the volume of the rectangular prism, use the formula $V = l \bullet w \bullet h$, and then substitute in the values for the length, width, and height. 8 inches • 3 inches • 10 inches = 240 inches^3.

D) 720 inches^3
Incorrect. Check your multiplication again. To find the volume of the rectangular prism, use the formula $V = l \bullet w \bullet h$ and then substitute in the values for the length, width, and height. The correct answer is 240 inches^3.

Self Check B

A machine takes a solid cylinder with a height of 9 mm and a diameter of 7 mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3 mm. Which of the following expressions would correctly find the volume of the solid?



A) $(\pi \cdot 7^2 \cdot 9) - (\pi \cdot 3^2 \cdot 9)$

B) $(\pi \cdot 3.5^2 \cdot 9) - (\pi \cdot 1.5^2 \cdot 9)$

C) $(\pi \cdot 7^2 \cdot 9) + (\pi \cdot 3^2 \cdot 9)$

D) $(\pi \cdot 3.5^2 \cdot 9) + (\pi \cdot 1.5^2 \cdot 9)$

A) $(\pi \cdot 7^2 \cdot 9) - (\pi \cdot 3^2 \cdot 9)$

Incorrect. Remember that the formula for the volume of a cylinder is $\pi r^2 h$, not $\pi d^2 h$. The correct answer is $(\pi \cdot 3.5^2 \cdot 9) - (\pi \cdot 1.5^2 \cdot 9)$.

B) $(\pi \cdot 3.5^2 \cdot 9) - (\pi \cdot 1.5^2 \cdot 9)$

Correct. You find the volume of the entire cylinder by multiplying $\pi \cdot 3.5^2 \cdot 9$, then subtract the empty cylinder in the middle, which is found by multiplying $\pi \cdot 1.5^2 \cdot 9$.

C) $(\pi \cdot 7^2 \cdot 9) + (\pi \cdot 3^2 \cdot 9)$

Incorrect. Remember that the formula for the volume of a cylinder is $\pi r^2 h$, not $\pi d^2 h$. Also, you should subtract the volume of the inner cylinder from the volume of the outer cylinder, not add it. The correct answer is $(\pi \cdot 3.5^2 \cdot 9) - (\pi \cdot 1.5^2 \cdot 9)$.

D) $(\pi \cdot 3.5^2 \cdot 9) + (\pi \cdot 1.5^2 \cdot 9)$

Incorrect. You have calculated the correct volumes for the outer and inner cylinders, but you should subtract the inner from the outer instead of adding them. The correct answer is $(\pi \cdot 3.5^2 \cdot 9) - (\pi \cdot 1.5^2 \cdot 9)$.

Unit Recap

1.1.1 Figures in 1 and 2 Dimensions

Geometric shapes and figures are all around us. A point is a zero-dimensional object that defines a specific location on a plane. A line is made up of an infinite number of points, all arranged next to each other in a straight pattern, and going on forever. A ray begins at one point and goes on towards infinity in one direction only. A plane can be described as a two-dimensional canvas that goes on forever.

When two rays share an endpoint, an angle is formed. Angles can be described as acute, right, obtuse, or straight, and are measured in degrees. You can use a protractor (a special math tool) to closely measure the size of any angle.

1.1.2 Properties of Angles

Parallel lines do not intersect, while perpendicular lines cross at a 90º angle. Two angles whose measurements add up to 180º are said to be supplementary, and two angles whose measurements add up to 90º are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

1.1.3 Triangles

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180º. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

1.1.4 The Pythagorean Theorem

The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle’s legs is the same as the square of the length of the triangle’s hypotenuse. This theorem is represented by the formula $a^2 + b^2 = c^2$. Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

1.2.1 Quadrilaterals

A quadrilateral is a mathematical name for a four-sided polygon. Parallelograms, squares, rectangles, and trapezoids are all examples of quadrilaterals. These quadrilaterals earn their distinction based on their properties, including the number of pairs of parallel sides they have and their angle and side measurements.

1.2.2 Perimeter and Area

The perimeter of a two-dimensional shape is the distance around the shape. It is found by adding up all the sides (as long as they are all the same unit). The area of a two-dimensional shape is found by counting the number of squares that cover the shape. Many formulas have been developed to quickly find the area of standard polygons, like triangles and parallelograms.

1.2.3 Circles

Circles are an important geometric shape. The distance around a circle is called the circumference, and the interior space of a circle is called the area. Calculating the circumference and area of a circle requires a number called pi (π), which is a non-terminating, non-repeating decimal. Pi is often approximated by the values 3.14 and $\frac{22}{7}$. You can find the perimeter or area of composite shapes—including shapes that contain circular sections—by applying the circumference and area formulas where appropriate.

1.3.1 Solids

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

Glossary

acute triangle	An angle measuring less than 90º.
angle	A figure formed by the joining of two rays with a common endpoint.
area	The amount of space inside a two-dimensional shape, measured in square units.
circumference	The distance around a circle, calculated by the formula $C = \pi d$.
complementary angles	Two angles whose measurements add up to 90º.
cone	A solid figure with a single circular base and a round, smooth face that diminishes to a single point.
congruent	Having the same size and shape.
corresponding angles	Angles of separate figures that are in the same position within each figure.
corresponding sides	Sides of separate figures that are opposite corresponding angles.
cube	A six-sided polyhedron that has congruent squares as faces.
cylinder	A solid figure with a pair of circular, parallel bases and a round, smooth face between them.
diameter	The length across a circle, passing through the center of the circle. A diameter is equal to the length of two radii.
equilateral triangle	A triangle with 3 equal sides. Equilateral triangles also have three angles that measure the same.
face	The flat surface of a solid figure.
hypotenuse	The side opposite the right angle in any right triangle. The hypotenuse is the longest side of any right triangle.
isosceles trapezoid	A trapezoid with one pair of parallel sides and another pair of opposite sides that are congruent.
isosceles triangle	A triangle with 2 equal sides and 2 equal angles.
leg, legs	In a right triangle, one of the two sides creating a right angle.

line	A line is a one-dimensional figure, which extends without end in two directions.
line segment	A finite section of a line between any two points that lie on the line.
obtuse angle, obtuse angles	An angle measuring more than 90º and less than 180º.
obtuse triangle	A triangle with one angle that measures between 90º and 180º.
parallel lines	Two or more lines that lie in the same plane but which never intersect.
parallelogram, parallelograms	A quadrilateral with two pairs of parallel sides.
perimeter	The distance around a two-dimensional shape.
perpendicular lines	Two lines that lie in the same plane and intersect at a 90º angle.
pi	The ratio of a circle's circumference to its diameter. Pi is denoted by the Greek letter π. It is often approximated as 3.14 or $\frac{22}{7}$.
plane	In geometry, a two-dimensional surface that continues infinitely. Any three individual points that don't lie on the same line will lie on exactly one plane.
point	A zero-dimensional object that defines a specific location on a plane. It is represented by a small dot.
polygon, polygons	A closed plane figure with three or more straight sides.
polyhedron, polyhedrons	A solid whose faces are polygons.
pyramid, pyramids	A polyhedron with a polygonal base and a collection of triangular faces that meet at a point.
Pythagorean Theorem	The formula that relates the lengths of the sides of any right triangle: $a^2 + b^2 = c^2$, where c is the hypotenuse, and *a* and *b* are the legs of the right triangle.
quadrilateral, quadrilaterals	A four-sided polygon.

radius	The distance from the center of a circle to any point on the circle.
ray	A half-line that begins at one point and goes on forever in one direction.
rectangle	A quadrilateral with two pairs of parallel sides and four right angles.
rectangular prism	A polyhedron that has three pairs of congruent, rectangular, parallel faces.
rhombus	A quadrilateral with four congruent sides.
right angle	An angle measuring exactly 90º.
right triangle, right triangles	A triangle containing a right angle.
scalene triangle	A triangle in which all three sides are a different length.
similar	Having the same shape but not necessarily the same size.
sphere	A solid, round figure where every point on the surface is the same distance from the center.
square	A quadrilateral whose sides are all congruent and which has four right angles.
straight angle	An angle measuring exactly 180º.
supplementary angles	Two angles whose measurements add up to 180º.
trapezoid	A quadrilateral with one pair of parallel sides.
triangle	A polygon with three sides.
vertex	A turning point in a graph. Also the endpoint of the two rays that form an angle.
volume	A measurement of how much it takes to fill up a three-dimensional figure. Volume is measured in cubic units.