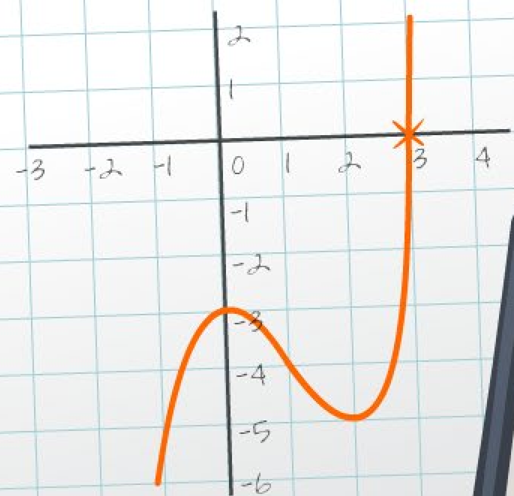
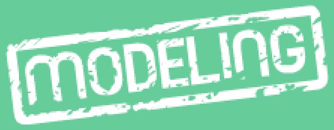
Rational Functions



Information

Common core icons

This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.



Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

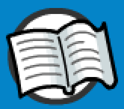
5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

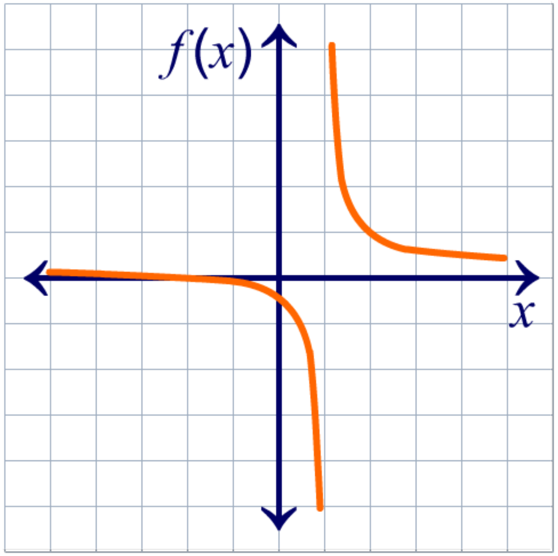
This icon indicates that the slide contains activities created in Flash. These activities are not editable.



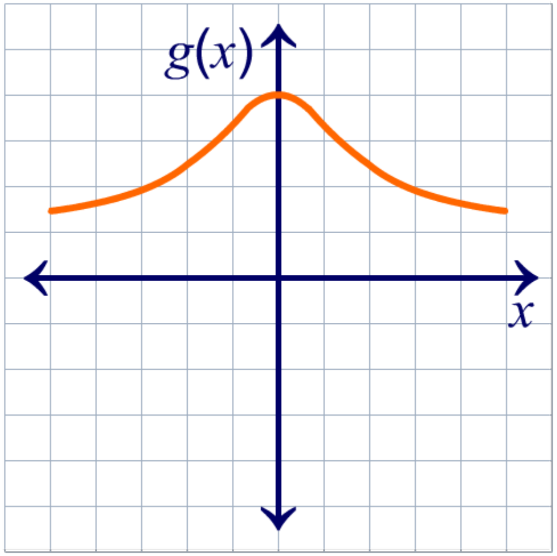
This icon indicates teacher's notes in the Notes field.

rational function: $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

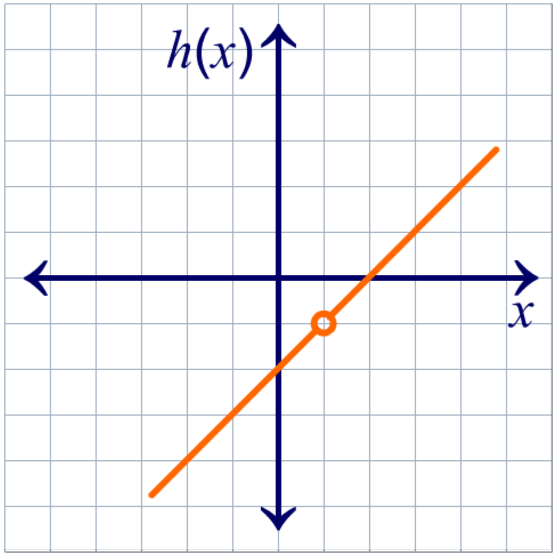
The graphs of rational functions vary greatly, for example:



$$f(x) = \frac{x+1}{3x-3}$$



$$g(x) = \frac{x^2+16}{x^2+4}$$

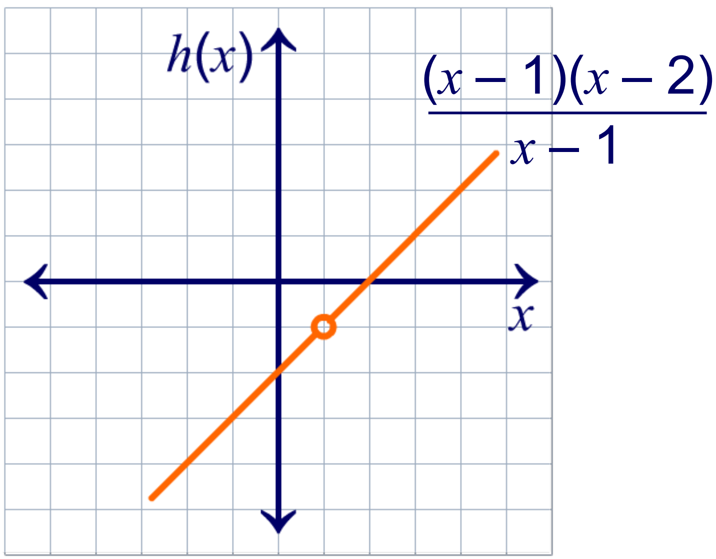


$$h(x) = \frac{(x-1)(x-2)}{x-1}$$

Discontinuity of rational functions

A function is **discontinuous** if it has jumps, breaks or holes. You could not draw the function without lifting your pencil.

If a is a real number for which the denominator of a rational function $h(x)$ is zero, then a is a **discontinuity** and not in the domain of $h(x)$.



A function is **continuous** if it has no discontinuities.

Why is the function $f(x) = \dfrac{x^2 + 16}{x^2 + 4}$ continuous?

If the denominator has no **real** zeros, then $f(x)$ has no discontinuities.

Continuous or discontinuous?

Drag the correct description of continuity into place

1) $\dfrac{x+3}{2}$

2) $\dfrac{1}{(x-2)^2}$

3) $\dfrac{(x+2)(x-2)}{x+2}$

4) $x^2 + 2x + 1$

continuous

discontinuous

In **expanded form**, the polynomials of the numerator and denominator are written in standard form.

rational function in expanded form:

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$$

Expanded form shows the order of the polynomials: m is the order of the numerator, and n is the order of the denominator.

Relate the function to the general expanded form:

$$f(x) = \frac{3x^3 + 16x^2 + 4}{-x^2 + x - 9}$$

numerator: The order is $m = 3$.
The coefficients are $a_0 = 4$, $a_1 = 0$, $a_2 = 16$, $a_3 = 3$.

denominator: The order is $n = 2$.
The coefficients are $b_0 = -9$, $b_1 = 1$, $b_2 = -1$.

In **factored form**, the polynomials are fully factored.

rational function in factored form:

$$f(x) = \frac{a_m(x - c_1)^{p_1}(x - c_2)^{p_2}\ldots(x - c_m)^{p_m}}{b_n(x - d_1)^{q_1}(x - d_2)^{q_2}\ldots(x - d_n)^{q_n}}$$

It shows the zeros (c_i) and multiplicities (p_i) of the numerator, and zeros (d_k) and multiplicities (q_k) of the denominator.

Relate the function to the general factored form:

$$f(x) = \frac{5x^2(x - 9)(x + 2)^4}{-4(x - 9)^3(x + 2)^4(x - 2)}$$

numerator: The zeros are $c_1 = 0$, $c_2 = 9$, and $c_3 = -2$ with multiplicities $p_1 = 2$, $p_2 = 1$ and $p_3 = 4$. The order is $m = 7$, and the coefficient $a_7 = 5$.

denominator: The zeros are $d_1 = 9$, $d_2 = -2$, and $d_3 = 2$ with multiplicities $q_1 = 3$, $q_2 = 4$ and $q_3 = 1$. The order is $n = 8$, and the coefficient $b_8 = -4$.

Finding vertical asymptotes

Vertical asymptotes of rational functions depend on the multiplicities of the zeros of the numerator and denominator.

To study vertical asymptotes, it is easiest to use factored form.

rational function in factored form:

$$f(x) = \frac{a_m(x-c_1)^{p_1}(x-c_2)^{p_2}\ldots(x-c_m)^{p_m}}{b_n(x-d_1)^{q_1}(x-d_2)^{q_2}\ldots(x-d_n)^{q_n}}$$

To find **vertical asymptotes**, use these criteria:

- If the numerator and denominator have no common zeros, then there is a vertical asymptote at every zero of the denominator: $x = d_1$, $x = d_2$, … $x = d_n$.
- If the numerator and denominator have a common zero, $c_i = d_k$, then there is a vertical asymptote at d_k if and only if $q_k > p_i$.

Holes occur when the numerator and denominator have a common zero and the multiplicities are equal.

The common terms can be reduced, so they do not affect the shape of the graph; however the function is still undefined at that point.

Find the vertical asymptotes and holes of the following function:

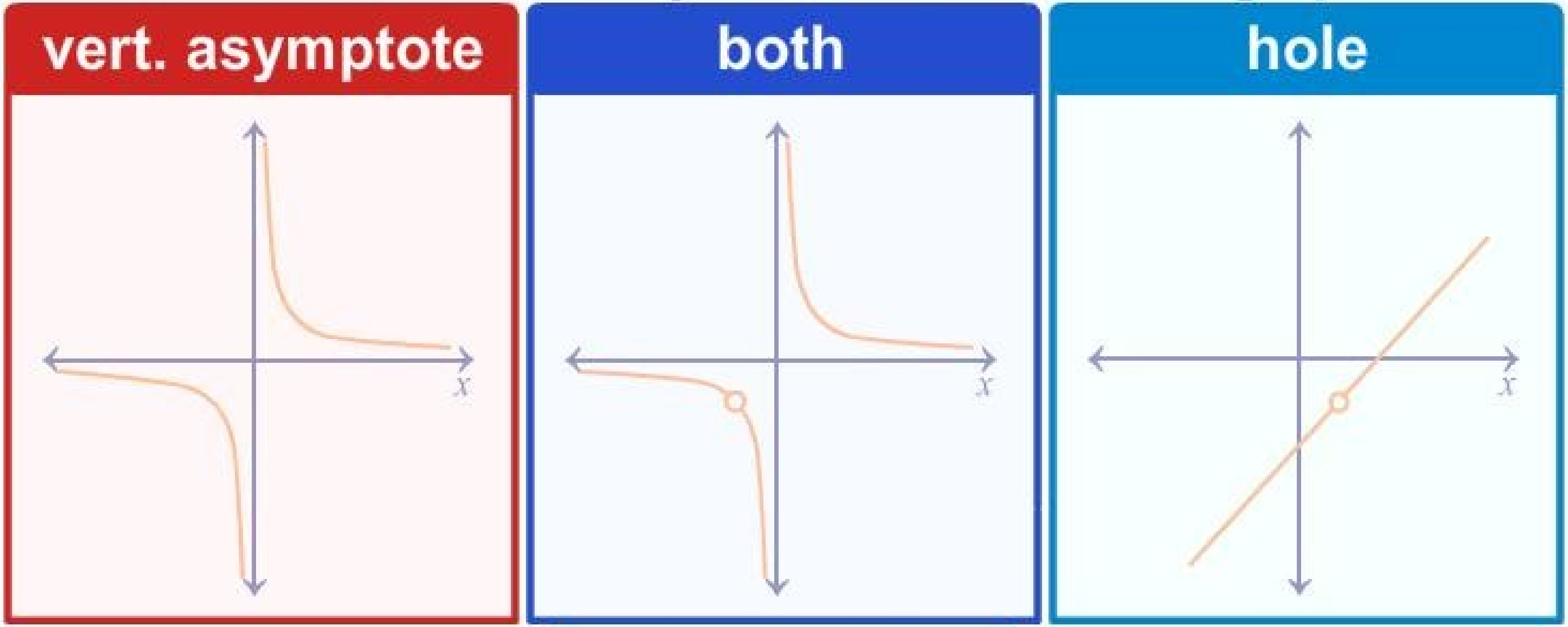
$$y = \frac{(x-1)(x-2)}{(x-1)(x-2)^2(x-3)}$$

The function is undefined at $x = 1$, $x = 2$ and $x = 3$.

There is a hole at x **= 1**.

There is a vertical asymptote at x **= 3**, and also at x **= 2** since the multiplicity of the zero at 2 is higher in the denominator than in the numerator.

Sort the functions by the features of their graphs



$$s(x) = \frac{x - 1}{(x - 1)(x - 2)}$$

Finding horizontal asymptotes

Horizontal asymptotes of rational functions depend on the degree of the numerator (m) and denominator (n).

To study horizontal asymptotes, use expanded form.

rational function in expanded form:

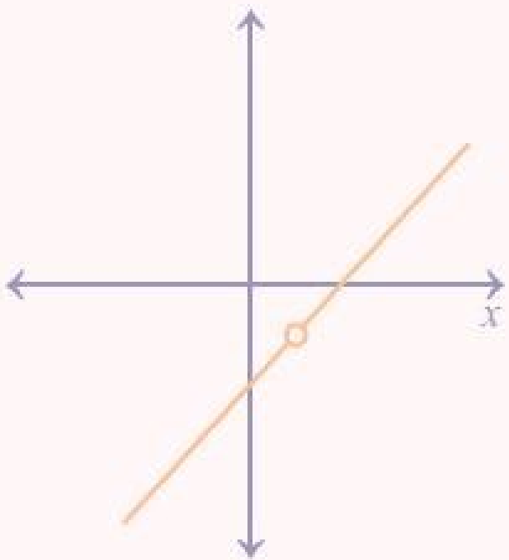
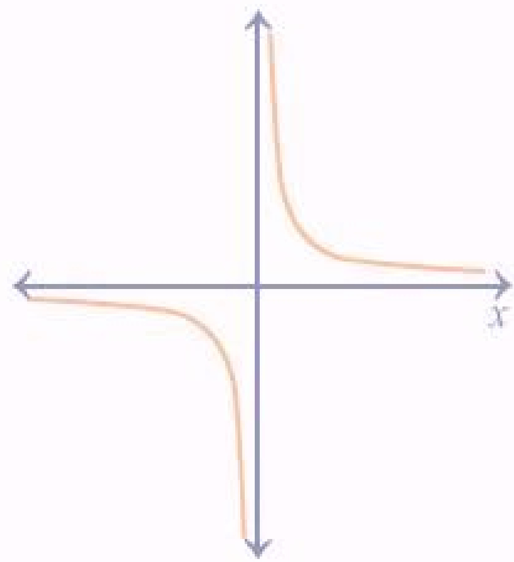
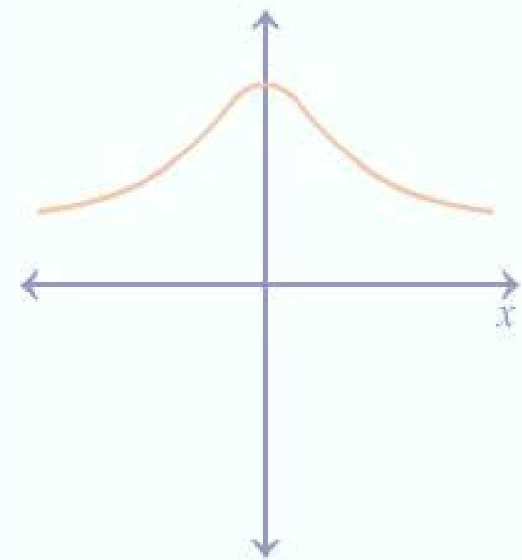
$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$$

To find **horizontal asymptotes**, use these criteria:

- If $m < n$, the graph has a horizontal asymptote at $y = 0$.
- If $m > n$, the graph has no horizontal asymptote.
- If $m = n$, the graph has a horizontal asymptote at $y = \frac{a_m}{b_n}$.

Horizontal asymptotes

Sort the functions by horizontal asymptotes

none	zero	non-zero
		

$$r(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

Match the functions to the features of their graphs

$$y = \frac{x^3(x-1)}{(x-1)(x-2)}$$

$$y = \frac{2(x-1)(x+2)}{2x(x-1)}$$

$$y = \frac{1}{(x-1)(x-2)}$$

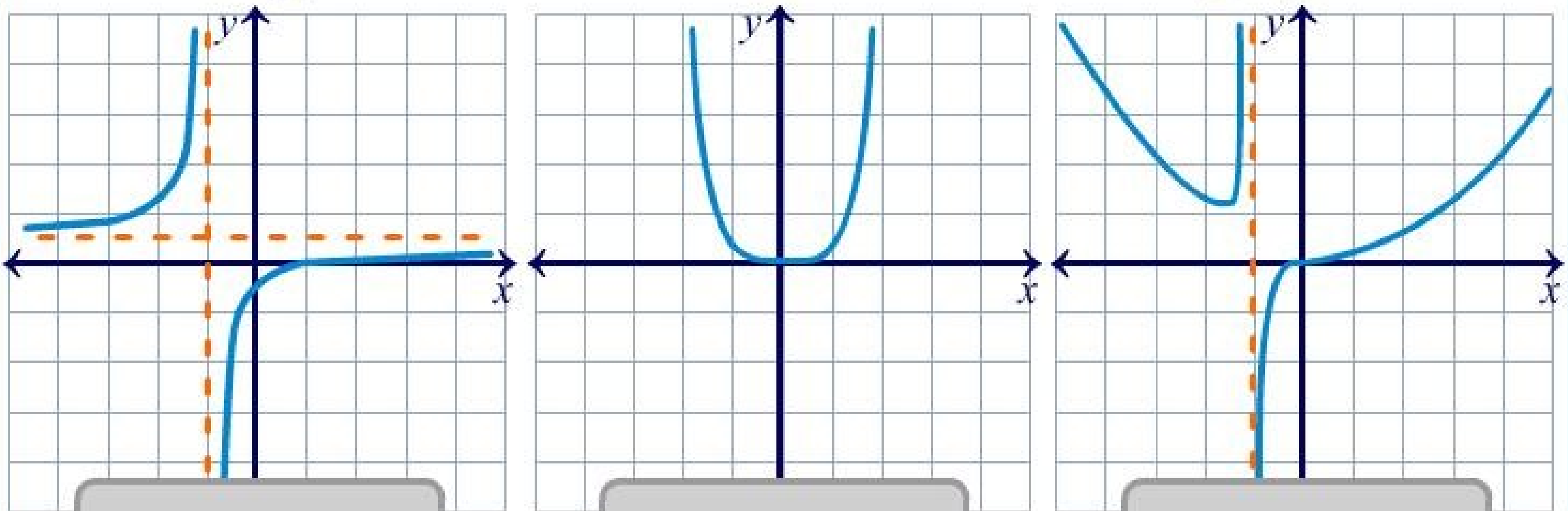
$$y = \frac{(x-1)(x+2)}{(x-1)(x^2+4)}$$

$$y = \frac{x(x-1)}{x-1}$$

vertical asymptotes	horizontal asymptotes	holes
$x = 0$	$y = 1$	$x = 1$
none	none	$x = 1$
none	$y = 0$	$x = 1$
$x = 1$, $x = 2$	$y = 0$	none
$x = 2$	none	$x = 1$

Match functions to graphs

Drag the correct functions to label the graphs



A

B

C

$$y = \frac{x-1}{2x+2}$$

$$y = \frac{x^3}{6(x+1)}$$

$$y = \frac{x^4}{3}$$