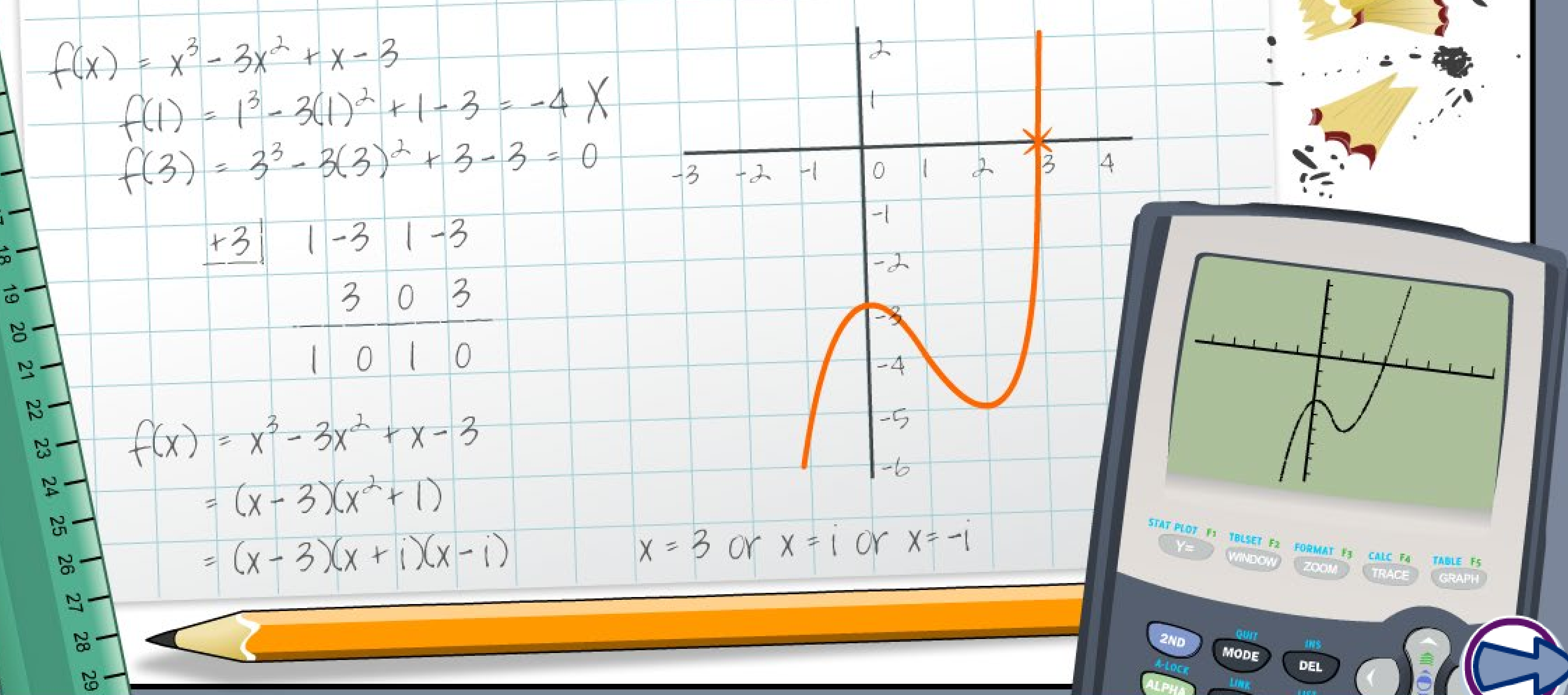
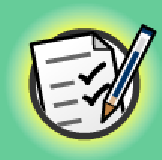
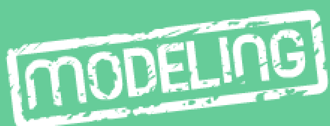
Dividing Polynomials



Common core icons



This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.



Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) **Make sense of problems and persevere in solving them.**
2) **Reason abstractly and quantitatively.**
3) **Construct viable arguments and critique the reasoning of others.**
4) **Model with mathematics.**
5) **Use appropriate tools strategically.**
6) **Attend to precision.**
7) **Look for and make use of structure.**
8) **Look for and express regularity in repeated reasoning.**

This icon indicates that the slide contains activities created in Flash. These activities are not editable.

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Dividing polynomials

Suppose we want to divide one polynomial $f(x)$ by another polynomial of lower order $g(x)$.

There are two possibilities. Either:

- $g(x)$ will leave a **remainder** when divided into $f(x)$.
- $g(x)$ will divide exactly into $f(x)$.
  In this case, $g(x)$ is a **factor** of $f(x)$ and the remainder is 0.

There are three methods to divide one polynomial by another. These are:

- **equating coefficients**
- using **long division**
- using **synthetic division**.

Equating coefficients (1)

Two polynomials are multiplied together and the resulting polynomial is $x^3 + x^2 - 10x + 8$.
One of the polynomials is $x + 4$. What is the other?

$x + 4$ is a linear polynomial. To obtain a cubic polynomial as required, multiply it by a quadratic of the form $ax^2 + bx + c$.

write out the multiplication: $(x + 4)(ax^2 + bx + c) = x^3 + x^2 - 10x + 8$

This is true for all values of x.

Since the expression on the left is equivalent to the expression on the right, the coefficients of x^3, x^2, x and the constant term must be the same on both sides of the equal sign.

Solve the equation by **equating coefficients**.

Equating coefficients (2)

Use the method of equating coefficients to find the values of a, b and c :

$$(x + 4)(ax^2 + bx + c) = x^3 + x^2 - 10x + 8$$

distribute: $ax^3 + bx^2 + cx + 4ax^2 + 4bx + 4c = x^3 + x^2 - 10x + 8$

group like terms: $ax^3 + (b + 4a)x^2 + (c + 4b)x + 4c = x^3 + x^2 - 10x + 8$

equate the coefficients of x^3: $\boldsymbol{a = 1}$

equate the coefficients of x^2: $b + 4a = 1$

substitute $a = 1$: $b + 4 = 1$

$\boldsymbol{b = -3}$

Use the method of equating coefficients to find the values of a, b and c :

$$(x + 4)(ax^2 + bx + c) = x^3 + x^2 - 10x + 8$$

$$ax^3 + (b + 4a)x^2 + (c + 4b)x + 4c = x^3 + x^2 - 10x + 8$$

already have: $a = 1$ $b = -3$

equate the coefficients of x: $c + 4b = -10$

substitute $b = -3$: $c - 12 = -10$

$c = 2$

substitute $a = 1$, $b = -3$ and $c = 2$ into the original equation:

$$(x + 4)(ax^2 + bx + c) = x^3 + x^2 - 10x + 8$$

$$(x + 4)(x^2 - 3x + 2) = x^3 + x^2 - 10x + 8$$

What is $x^3 + x^2 - 10x + 8$ divided by $x + 4$?

Sometimes a polynomial leaves a **remainder** when it divides into another polynomial.

What is $f(x) = 3x^2 + 11x - 8$ divided by $x + 5$?

write $f(x) = 3x^2 + 11x - 8$ in terms of a quotient and a remainder:

$$3x^2 + 11x - 8 = (x + 5)(\text{quotient}) + (\text{remainder})$$

Since the dividend is a quadratic polynomial, the quotient must be a linear polynomial of the form $ax + b$.

write: $3x^2 + 11x - 8 = (x + 5)(ax + b) + r$

expand: $3x^2 + 11x - 8 = ax^2 + bx + 5ax + 5b + r$

group like terms: $3x^2 + 11x - 8 = ax^2 + (b + 5a)x + 5b + r$

Remainders (2)

What is $f(x) = 3x^2 + 11x - 8$ divided by $x + 5$?

$$3x^2 + 11x - 8 = ax^2 + (b + 5a)x + 5b + r$$

coefficients of x^2: $\boldsymbol{a = 3}$

coefficients of x: $b + 5a = 11$

substitute $a = 3$: $b + 15 = 11$

$\boldsymbol{b = -4}$

constants: $5b + r = -8$

substitute $b = -4$: $-20 + r = -8$

$\boldsymbol{r = 12}$

substitute these values into the original equation:

$$3x^2 + 11x - 8 = (x + 5)(ax + b) + r$$

$$3x^2 + 11x - 8 = (x + 5)(3x - 4) + 12$$

so $3x^2 + 11x - 8$ divided by $x + 5$ is **$3x - 4$ remainder 12**

or $\boldsymbol{3x - 4 + \dfrac{12}{x + 5}}$

Long division of numbers

The method of **long division** used for numbers can be applied to the division of polynomial functions.

Let's start by looking at the method for numbers.

Divide 5482 by 15 using long division.

```
      3 6 5
   ┌────────
15 ) 5 4 8 2
   – 4 5
     ───
       9 8
     – 9 0
       ───
         8 2
       – 7 5
         ───
           7
```

This shows that 5482 contains **365** lots of 15, leaving a remainder of **7**.

write this as an equation:

5482 ÷ 15 = 365 remainder 7

or $5482 = 15 \times 365 + 7$

dividend = divisor × quotient + remainder

Long division of polynomials

We can use the same method of long division to divide polynomials.

What is $f(x) = x^3 - x^2 + 5x - 2$ divided by $x + 1$?

$$
\begin{array}{rl}
 & \mathbf{x^2 - 2x + 7} \\
x + 1 & \overline{)\,x^3 - x^2 + 5x - 2} \\
 & -(x^3 + x^2) \\
\hline
 & -2x^2 + 5x \\
 & -(-2x^2 - 2x) \\
\hline
 & 7x - 2 \\
 & -(7x + 7) \\
\hline
 & \mathbf{-9}
\end{array}
$$

This shows that $x^3 - x^2 + 5x - 2$ divided by $x + 1$ is $\mathbf{x^2 - 2x + 7}$, remainder –9

Since the remainder is –9, $x + 1$ is not a factor of $f(x)$.

write this as an equation:

$$\frac{x^3 - x^2 + 5x - 2}{x + 1} = \mathbf{x^2 - 2x + 7 - \frac{9}{x + 1}}$$

or

$$x^3 - x^2 - 7x + 3 = (x - 3)(\mathbf{x^2 + 2x + 7}) - 9$$

Long division practice

1. What is $f(x) = x^3 - x^2 - 7x + 3$ divided by $x - 3$?
2. What is $f(x) = 2x^3 + x^2 + 8$ divided by $x + 2$?
3. What is $f(x) = x^3 - 3x^2 + 2x$ divided by $x - 1$?
4. What is $f(x) = x^3 + 6x^2 + 13x + 7$ divided by $x + 2$?
5. What is $f(x) = 3x^3 - 5x + 50$ divided by $x - 3$?

Shortening the process

Notice all the duplication of information in polynomial long division.

Some expressions are repeated.

The powers of x follow a consistent pattern, too, regardless of the division being performed.

Because of this, we can shorten the process of long division with a technique called **synthetic division**.

$$
\begin{array}{r}
2x^2 + x + 2 \\
x-2\,\overline{)\,2x^3 - 3x^2 + 0x + 1} \\
2x^3 - 4x^2 \\ \hline
x^2 + 0x \\
x^2 - 2x \\ \hline
2x + 1 \\
2x - 4 \\ \hline
5
\end{array}
$$

What is $f(x) = 2x^3 - 5x^2 + x - 7$ divided by $x - 3$?

by long division

$$
\begin{array}{r|l}
 & 2x^2 + x + 4 \\
\hline
x - 3 & 2x^3 - 5x^2 + 1x - 7 \\
 & \underline{2x^3 - 6x^2} \\
 & \qquad\quad x^2 + 1x \\
 & \qquad\quad \underline{x^2 - 3x} \\
 & \qquad\qquad\quad 4x - 7 \\
 & \qquad\qquad\quad \underline{4x - 12} \\
 & \qquad\qquad\qquad\quad 5
\end{array}
$$

by synthetic division

$$
\begin{array}{r|rrrr}
3 & 2 & -5 & 1 & -7 \\
 & & 6 & 3 & 12 \\
\hline
 & 2 & 1 & 4 & 5
\end{array}
$$

- 2: coefficient of x^2
- 1: coefficient of x
- 4: constant
- 5: remainder

Synthetic division (2)

Synthetic division is quicker than long division and can be easier because it uses addition rather than subtraction.

What is $f(x) = 2x^3 - 5x^2 + x - 7$ divided by $x - 3$?

3	2	–5	1	–7
		6	3	12
	2	1	4	5

Write the coefficients of the dividend in a row. Include zeros, as in long division.

To divide by $x - c$, put c on "the shelf".

Bring down the first coefficient. Multiply the number on the shelf, 3, by the first number below the line, 2.

Write the product, 6, in the next column. Add it to the number above it and write the answer, 1, below the line.

Multiply this number by the number on the shelf, and repeat.

Synthetic division example

Remember to reverse the sign of the constant in the divisor and add the rows together.

What is $3x^4 + 13x^3 + 54x + 17$ divided by $x + 5$?

–5	3	13	0	54	17
		–15	10	–50	–20
	3	–2	10	4	–3

$$3x^3 - 2x^2 + 10x + 4 - \frac{3}{x + 5}$$

Notice the –5 on the shelf, since the divisor is $(x + 5)$ and $x = -5$ is a zero of this divisor.

There is also a minus sign in front of the fraction, since the remainder is negative.

Practicing synthetic division

Synthetic division

Drag the numbers into place to complete the synthetic division.

What is $x^3 - x^2 - 7x + 3$ divided by $x - 3$?

?	?	?	?	?
		?	?	?
	?	?	?	?

? x^2 + ? x – ?

	1	2	3	4
0	–1	–2	–3	–4
5	6	7	8	9
–5	–6	–7	–8	–9

S C ↺ ?

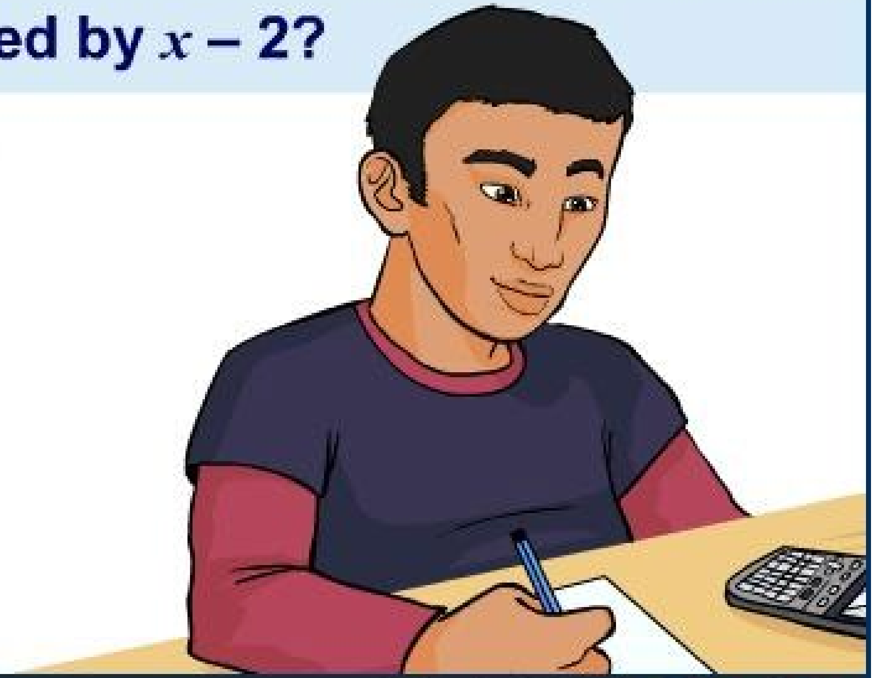
Comparing methods of dividing polynomials

There are three different ways of dividing polynomials. Use this activity to see all three methods applied to the same polynomial.

What is $f(x) = 2x^3 - 3x^2 + 1$ divided by $x - 2$?

- view equating coefficients
- view long division
- view synthetic division

reveal solution



Remainder theorem (1)

What is $f(x) = x^3 - 10x^2 + 22x - 5$ divided by $x - 7$?

$$
\begin{array}{r|l}
 & x^2 - 3x + 1 \\
\hline
x - 7 & x^3 - 10x^2 + 22x - 5 \\
 & x^3 - 7x^2 \\
\hline
 & -3x^2 + 22x \\
 & -3x^2 + 21x \\
\hline
 & x - 5 \\
 & x - 7 \\
\hline
 & 2
\end{array}
$$

What is $f(7)$?

$$f(7) = 7^3 - 10(7)^2 + 22(7) - 5$$
$$= 343 - 490 + 154 - 5$$
$$= 2$$

Why is $f(7)$ the same as the remainder when $f(x) = x^3 - 10x^2 + 22x - 5$ is divided by $x - 7$?

Remainder theorem (2)

Why is $f(7)$ the same as the remainder when $f(x) = x^3 - 10x^2 + 22x - 5$ is divided by $x - 7$?

$$f(x) = \text{(divisor)(quotient)} + \text{(remainder)}$$

$$f(x) = (x - 7)\text{(quotient)} + \text{(remainder)}$$

$$f(7) = (7 - 7)\text{(quotient)} + \text{(remainder)}$$

$$f(7) = 0 + \text{(remainder)}$$

So when $f(x)$ is divided by $(x - 7)$, the remainder is equal to $f(7)$.

The same argument applies to any polynomial $f(x)$ and any divisor $(x - a)$.

The remainder theorem: When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

Using the remainder theorem

Find the remainder when...

1. ...$f(x) = x^3 - 4x^2 + 7x - 8$ is divided by $(x - 2)$.
2. ...$f(x) = x^3 - 3x^2 - 8x + 5$ is divided by $(x + 2)$.
3. ...$f(x) = x^2 + 6x + 6$ is divided by $(x + 3)$.
4. ...$f(x) = x^3 - 3x^2 - 7x - 11$ is divided by $(x - 5)$.
5. ...$f(x) = x^4 + x^3 + 6x^2 - 7x$ is divided by $(x - 1)$.

Find the remainder when the polynomial $f(x) = x^3 + 5x^2 + 10x + 12$ is divided by $(x + 3)$.

By the remainder theorem, the remainder is $f(-3)$.

$$f(x) = x^3 + 5x^2 + 10x + 12$$

substitute: $f(-3) = (-3)^3 + 5(-3)^2 + 10(-3) + 12$

evaluate: $f(-3) = -27 + 45 - 30 + 12$

$$f(-3) = 0$$

What can you conclude about $(x + 3)$?

$(x + 3)$ is a factor of $f(x)$.

The factor theorem: $(x - a)$ is a factor of a polynomial $f(x)$ if and only if $f(a) = 0$.

How can we find the remainder when a polynomial $f(x)$ is divided by an expression of the form $(ax - b)$?

$$f(x) = (ax - b)(\text{quotient}) + (\text{remainder})$$

The remainder is equal to $f(x)$ at a value of x such that $ax - b$ is zero.

let: $ax - b = 0$

rearrange for x: $ax = b$

$$x = \frac{b}{a}$$

Therefore the remainder is given by $f\left(\frac{b}{a}\right)$.

The remainder and factor theorems:

when a polynomial $f(x)$ is divided by $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$. Therefore, $(ax - b)$ is a factor of $f(x)$ if and only if $f\left(\frac{b}{a}\right) = 0$.

Finding the remainder

Match the polynomials and divisors to their remainders

$f(x) = -2x^3 + 11x^2 - 11x - 4$ divided by $(2x - 5)$	-2
$f(x) = 2x^3 + 5x^2 + 6x + 2$ divided by $(2x + 1)$	-3
$f(x) = 3x^3 - 22x^2 + 4x + 19$ divided by $(x - 7)$	0
$f(x) = 8x^3 + 8x^2 - 8x - 11$ divided by $(4x + 4)$	6
$f(x) = -x^3 + 4x^2 - 2x + 2$ divided by $(3 - x)$	5