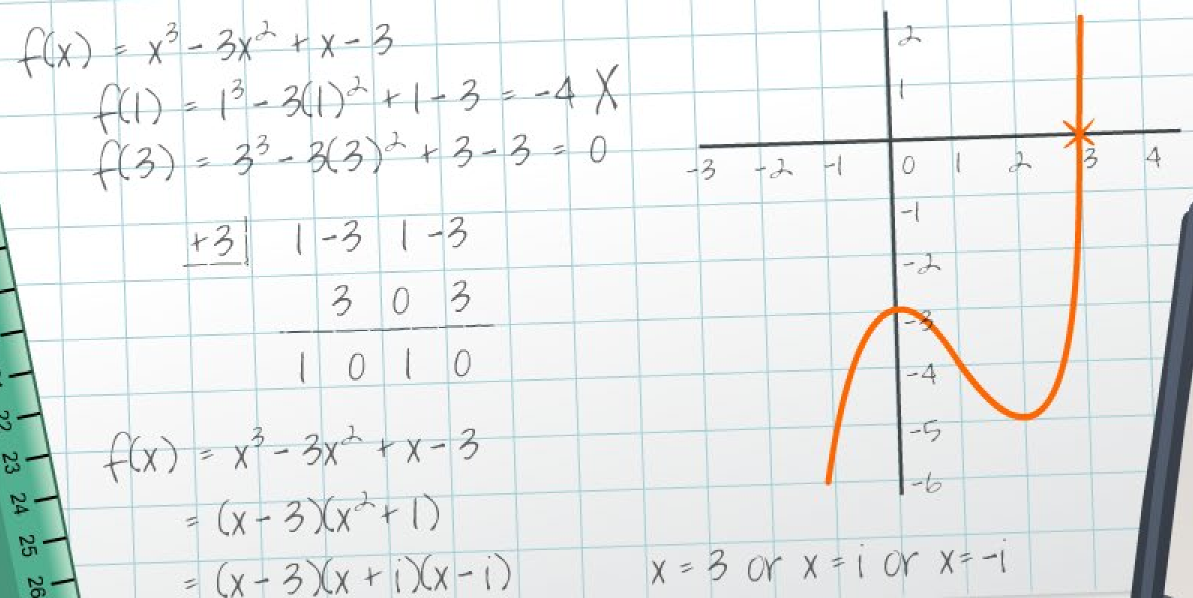
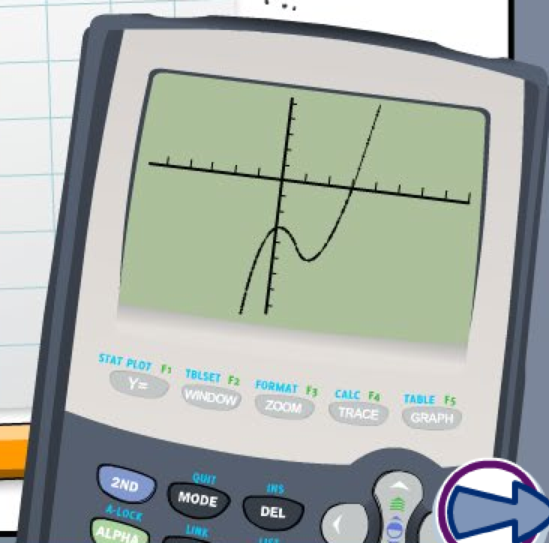
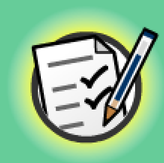
Complex Numbers





Common core icons



This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.

Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

This icon indicates that the slide contains activities created in Flash. These activities are not editable.

This icon indicates teacher's notes in the Notes field.

Review of number sets

Do you know how to classify different numbers?

natural numbers

integers

Press on each type of number to reveal a description and some examples.

rational numbers

irrational numbers

Solutions to equations

The solution to the equation $x + 5 = 2$ is an **integer**.

The solution to the equation $2x = 5$ is a **rational number**.

The solution to the equation $x^2 = 5$ is an **irrational number**.

What kind of number is the solution to $x^2 = –1$?

This equation asks:
“What number squared gives –1?”

The square root of a negative number is not a real number.

To solve an equation like $x^2 = –1$,
a new set of numbers is introduced.

The square root of a negative number is defined using ***i***.

The imaginary unit *i*: $i^2 = -1$

The solution to the equation $x^2 = -1$ is therefore $x = \pm i$.

a) Find: i) $\sqrt{(-4)}$ ii) $\sqrt{(-7)}$ iii) $\sqrt{(-12)}$

i) $\sqrt{(-4)}$ is the same as $\sqrt{(-1)} \times \sqrt{4} = i \times \sqrt{4} = \mathbf{2i}$

ii) $\sqrt{(-7)}$ is the same as $\sqrt{(-1)} \times \sqrt{7} = i \times \sqrt{7} = \mathbf{i\sqrt{7}}$

iii) $\sqrt{(-12)}$ is the same as $\sqrt{(-1)} \times \sqrt{12} = i \times 2\sqrt{3} = \mathbf{2i\sqrt{3}}$

$\sqrt{-a} = i\sqrt{a}$ for any positive real number, a

Finding powers of i

Look at these results:

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \times i = -i$$

$$i^4 = i^2 \times i^2 = (-1)^2 = 1$$

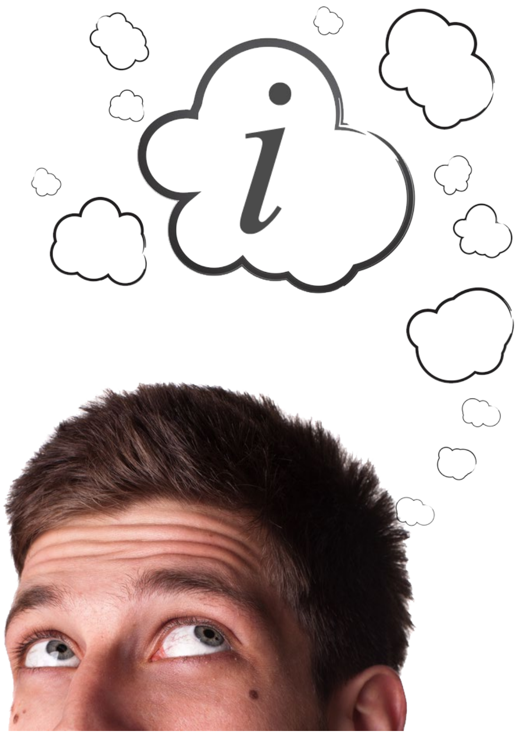
$$i^5 = i^4 \times i = 1 \times i = i$$

Write an explanation for finding i to any power and use it to complete the answers below.

Taking the set of real numbers and joining it with imaginary numbers like $2i$, $–5i$, and $2i\sqrt{3}$, gives a new set of numbers.

Complex numbers: numbers in the form $a + bi$

where a and b are real numbers and i is the imaginary unit



What kind of number is $a + bi$ when:

i) $a = 0$

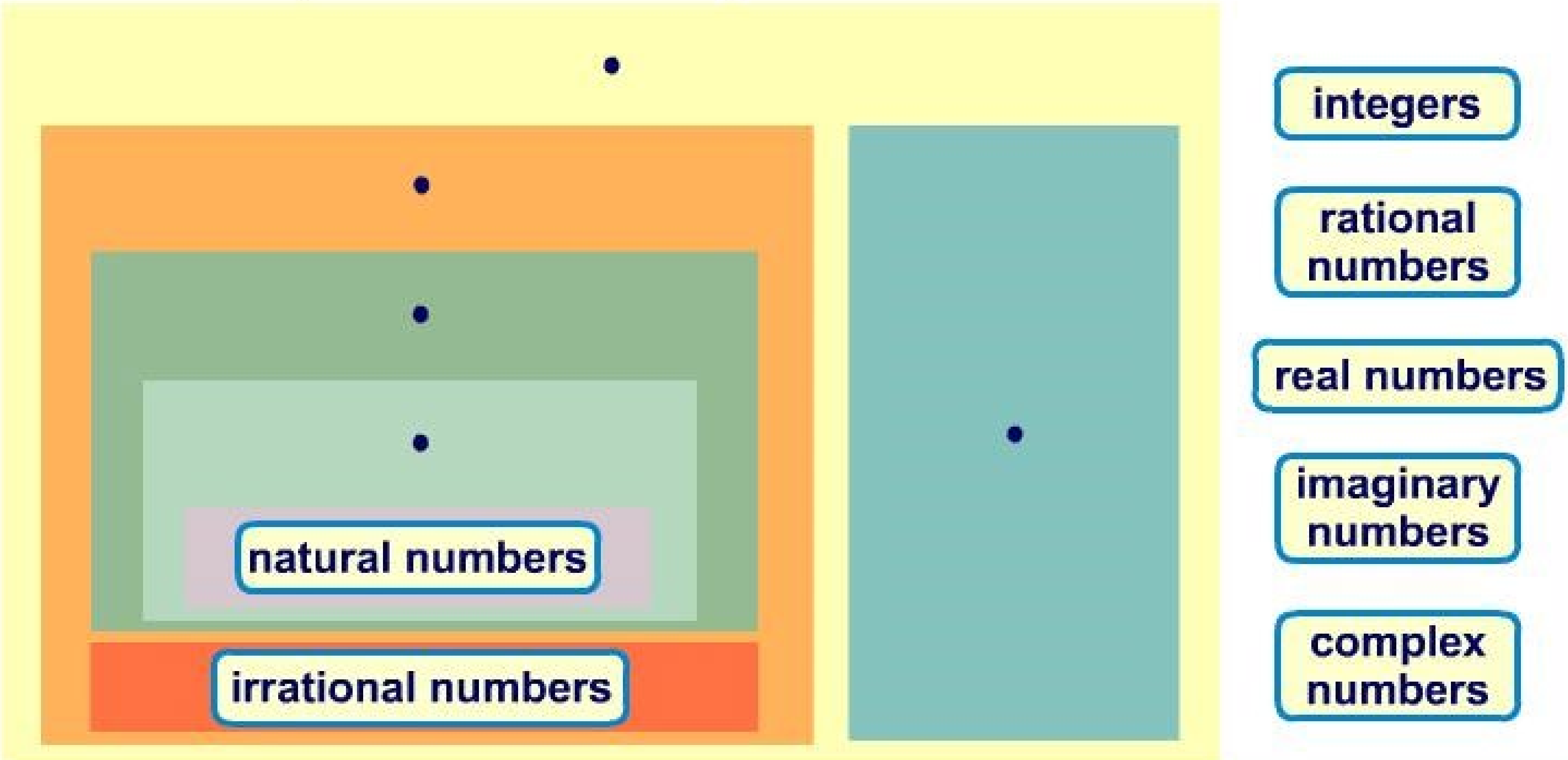
ii) $b = 0$?

i) If $a = 0$ then the number is **imaginary**: bi.

ii) If $b = 0$ then the number is **real**: a.

Number sets Venn diagram

Complete the Venn diagram of the sets of numbers.



Equality of complex numbers

Two complex numbers $(a + bi)$ and $(c + di)$ are said to be equal if and only if $a = c$ and $b = d$.

If $-7 + 3i = (2x + 3) + (4y - 5)i$, find the value of x and y.

By the equality of complex numbers two equations can be set up: one equating real parts and one equating imaginary parts.

equate real parts: $-7 = 2x + 3$

solve: $\mathbf{x = -5}$

equate imaginary parts: $4y - 5 = 3$

solve: $\mathbf{y = 2}$

Check this by substituting in $x = -5$ and $y = 2$ into the right-hand side to see if it is equal to the complex number on the left hand side.

$(2(-5) + 3) + (4(2) - 5)i = -7 + 3i$ ✓

Operations with complex numbers

Addition of complex numbers: **What is $(a + bi) + (c + di)$?**

answer

Subtraction of complex numbers: **What is $(a + bi) - (c + di)$?**

answer

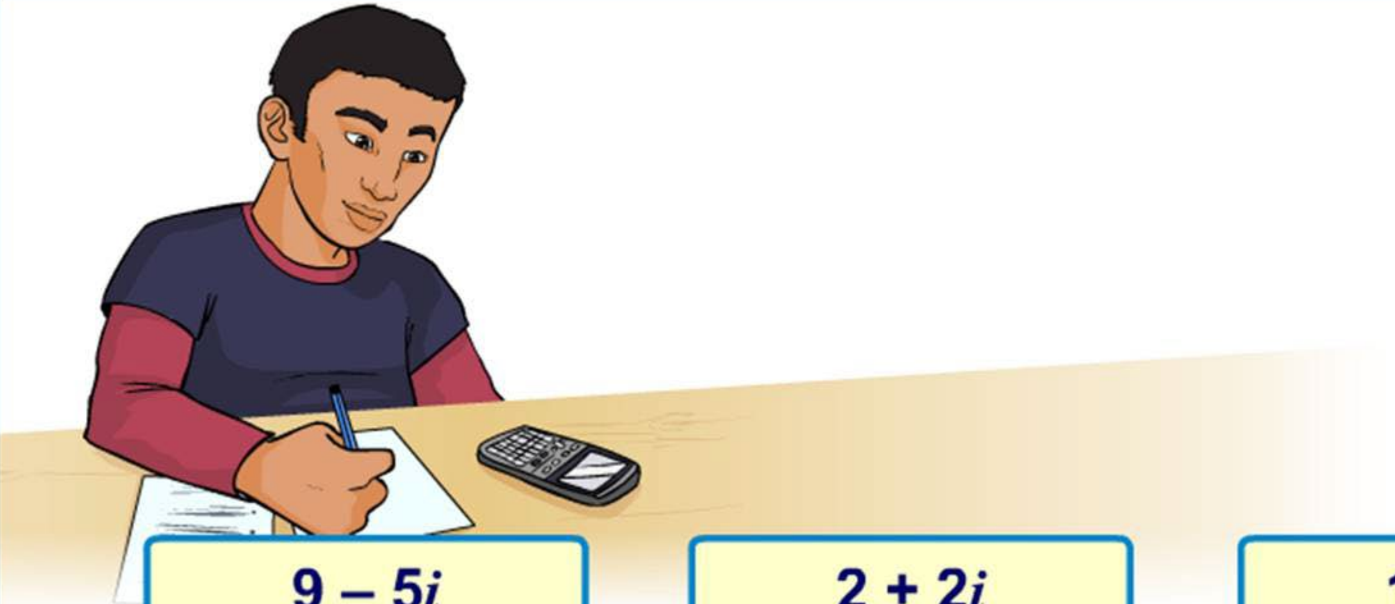
Multiplication of complex numbers: **What is $(a + bi)(c + di)$?**

answer

Operations with complex numbers

Question: 1/4

If $z_1 = 4 - 3i$ and $z_2 = -2 + 5i$, find $z_1 + z_2$.



$9 - 5i$

$2 + 2i$

$1 + 3i$

A special product

Find the product $(a + bi)(a - bi)$.

distribute:	$a^2 - abi + abi - b^2i^2$
$i^2 = -1$:	$a^2 - b^2(-1)$
	$a^2 + b^2$

What kind of number is $a^2 + b^2$?

Since there is no "i" (imaginary part), it is a **real number**.

The complex numbers $a + bi$ and $a - bi$ are a special pair of complex numbers known as **complex conjugates**.

product of complex conjugates:
$(a + bi)(a - bi) = a^2 + b^2$

Complex conjugates

complex conjugate of a complex number $z = a + bi$:

$$\bar{z} = a - bi$$

If $z = a + bi$ and its conjugate is $\bar{z} = a - bi$ find the following and describe the kind of number the answer is:

i) $z + \bar{z}$ **ii) $z - \bar{z}$**

i) $(a + bi) + (a - bi)$

add: $= 2a + bi - bi$

simplify: $= \mathbf{2a}$ which is a **real number**

i) $(a + bi) - (a - bi)$

subtract: $= a - a + 2bi$

simplify: $= \mathbf{2bi}$ which is a **pure imaginary number** (as long as $b \neq 0$)

Complex conjugates

Find the complex conjugate of:

1) $z = \frac{1}{2} + \frac{1}{4}i$?

2) $z = -3 - 4i$?

3) $z = 2i$?

4) $z = 3$?

5) $z = -2bi + 5a$?

Discuss the following in groups:

a) What is the additive identity and multiplicative identity for the set of real numbers?

b) Identify the additive identity and multiplicative identity in the complex number system.

c) How could you set up equations to find the additive inverse of $(-6 + 3i)$?

Remember, additive inverses must give the additive identity when we add them.

Additive and multiplicative identity

a) In the real number system:

- the additive identity is **0** (anything plus 0 equals itself)
- the multiplicative identity is **1** (anything multiplied by 1 equals itself).

b) In the complex number system:

- the additive identity is $\mathbf{0 + 0}\boldsymbol{i}$, e.g. $(2 + 2i) + (0 + 0i) = 2 + 2i$
- the multiplicative identity is $\mathbf{1 + 0}\boldsymbol{i}$, e.g. $(2 + 2i)(1 + 0i) = 2 + 0i + 2i + 0$
  $= 2 + 2i$

c) To find the additive inverse, $a + bi$, of $-6 + 3i$, write:

$(-6 + 3i) + (a + bi) = (0 + 0i)$

equate real parts: $-6 + a = 0$
$a = 6$

equate imaginary parts: $3 + b = 0$
$b = -3$

So the additive inverse of $-6 + 3i$ is $\mathbf{6 - 3}\boldsymbol{i}$.

Solving quadratic equations

Solve the quadratic equation $x^2 = -8$.

square root:	$x = \pm\sqrt{(-8)}$
product rule:	$x = \pm\sqrt{8} \times \pm\sqrt{(-1)}$
$\sqrt{-1} = i$:	$x = \pm\,(2\sqrt{2} \times i)$
identify solutions:	$\mathbf{x = 2\sqrt{2}\,i}$ or $\mathbf{x = -2\sqrt{2}\,i}$

When a quadratic equation like this one has no real roots, it has **complex roots**.

Recall the quadratic formula used to find the roots of $ax^2 + bx + c$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In what circumstance will this formula give complex roots?

Solving quadratic equations

Solve $x^2 + 4x + 9 = 0$ using the quadratic formula.

recall formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1$, $b = 4$ and $c = 9$.

substitute a, b, c: $x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(9)}}{2(1)}$

simplify: $x = \dfrac{-4 \pm \sqrt{-20}}{2}$

$\sqrt{-20} = i\sqrt{20}$: $x = \dfrac{-4 \pm 2i\sqrt{5}}{2}$

identify solutions: $\mathbf{x = -2 + i\sqrt{5}}$ or $\mathbf{x = -2 - i\sqrt{5}}$

What do you notice about the two solutions?

The solutions are a pair of complex conjugates.

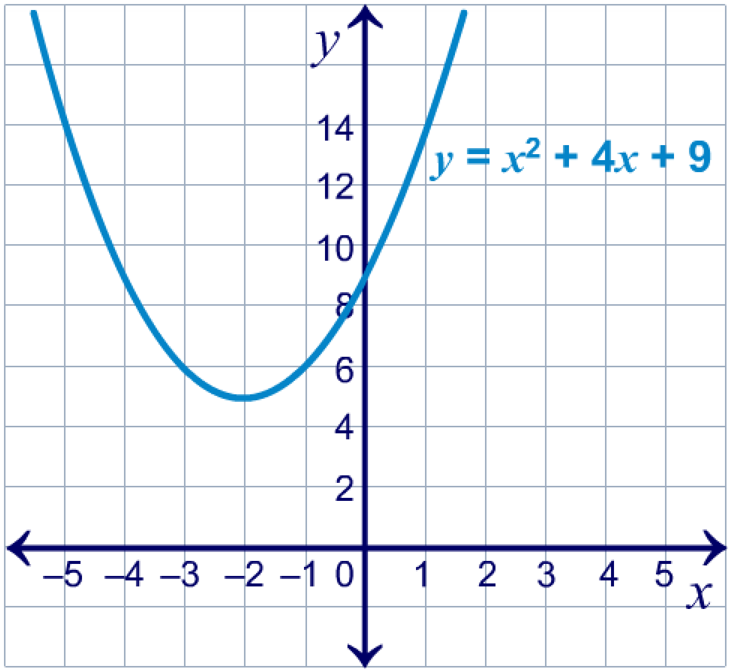
Complex roots on graphs

The real roots of a quadratic function are the x-intercepts of the curve.

What do you think is the distinctive feature of the graph of a function with complex roots?

The graph of a function with no real roots (i.e. complex roots) has no x-intercepts.

For example, look at the graph of $f(x) = x^2 + 4x + 9$.



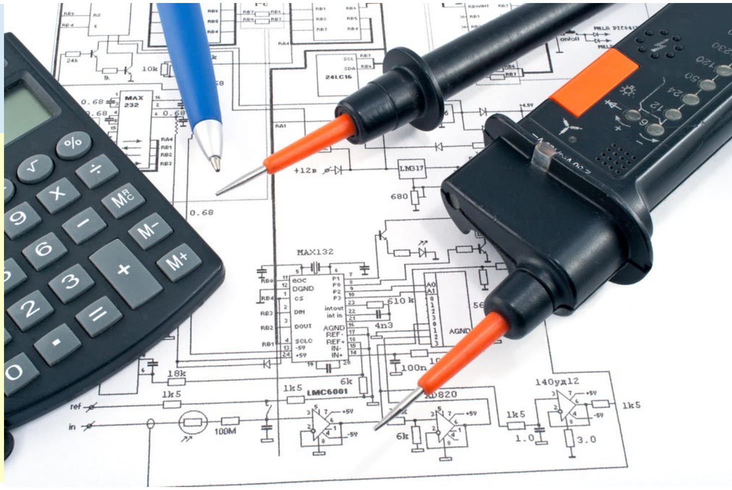
Complex numbers in electronics

Engineers often need to work with complex numbers in their calculation.

Bill, an electrical technician, uses the formula $V = IZ$ at work. V is the voltage, I is current and Z is impedance. He needs to find the voltage at a specific point and is able to determine that the circuit has a current flowing through it that was $5 + 2i$ amps and the impedance was $4 - i$ ohms.

Use the formula to find the voltage.

	$V = IZ$
substitute:	$V = (5 + 2i)(4 - i)$
distribute:	$V = 20 - 5i + 8i - 2i^2$
$i^2 = -1$:	$V =$ **$22 + 3i$ volts**



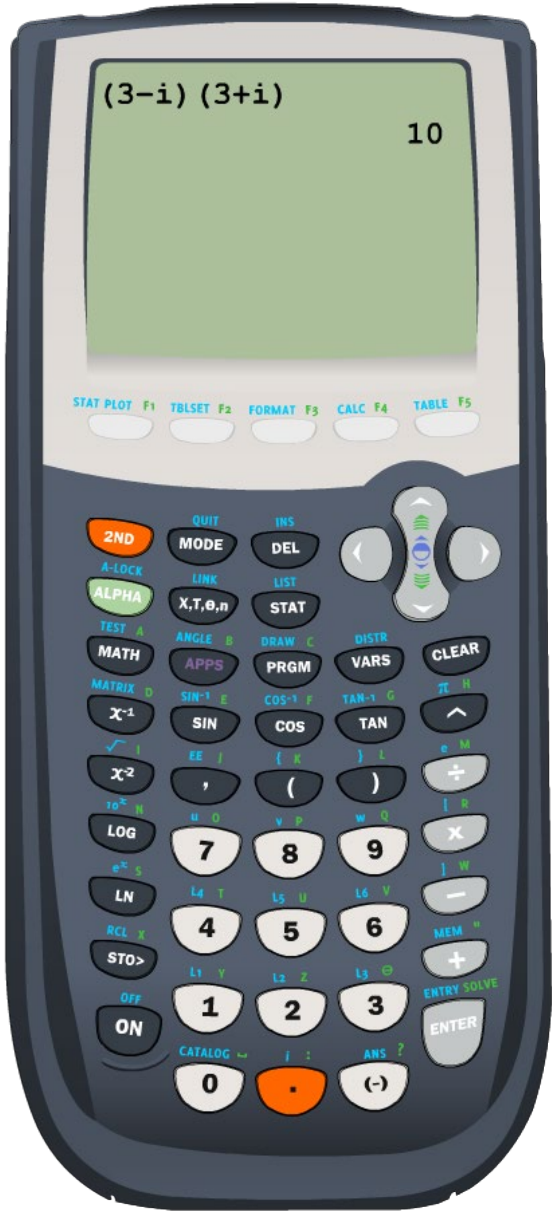
Checking using a calculator

Your graphing calculator can be used to perform calculations with complex numbers.

Press "MODE" and scroll down to "$a + bi$". Press "ENTER" to select this mode.

Return to the calculation screen and enter the calculation you wish to check.

On the TI-83/4, the imaginary unit "i" can be entered by pressing the "2nd" button then the "." button.



Complex numbers quiz

Simplify each of the following:

1) $(2i)(-3i)$?

2) $(0.6 - 0.2i) - (1.7 - 5i)$?

3) $(3 - i)(3 + i)$?

4) i^{13} ?

5) $(2i - 4)(\frac{1}{2} + i) + 7i$?