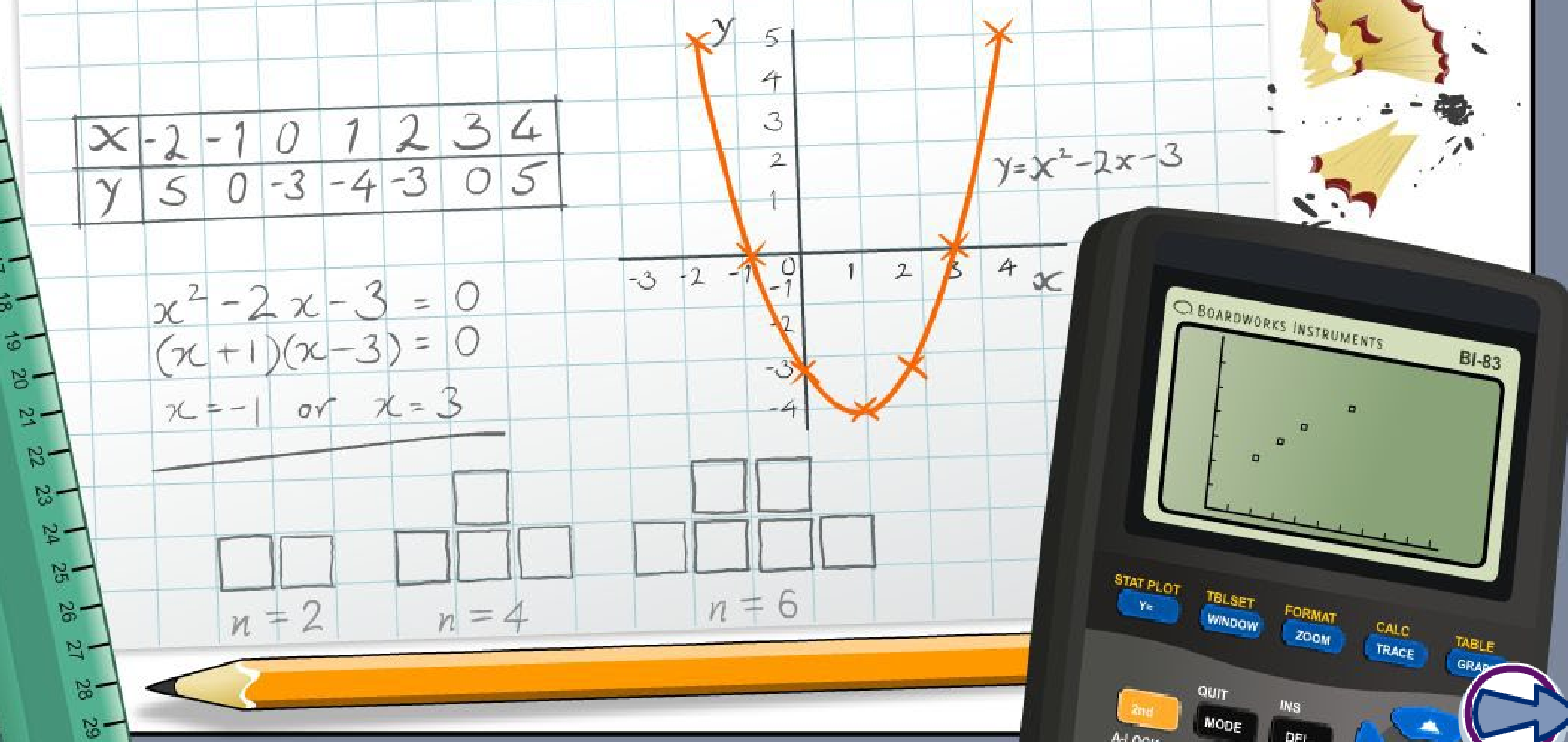
Systems of linear and quadratic equations



Information

Common core icons

This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.

Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

They are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

This icon indicates that the slide contains activities created in Flash. These activities are not editable.

This icon indicates teacher's notes in the Notes field.

A line and a parabola

Suppose one of the equations in a system of equations is **linear** and the other is a **quadratic** equation of the form $y = ax^2 + bx + c$.

Substitution

When one equation in a system of equations is quadratic, we often solve them by substitution.

Solve: $y = x^2 + 1$ ①

$y = x + 3$ ②

Substituting equation ① into equation ② gives $x^2 + 1 = x + 3$

Collect all the terms onto the left-hand side so that we can factor and use the Zero Product Property:

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$x = \mathbf{-1}$ or $x = \mathbf{2}$

Substituting into equations

We can then substitute these values of x into one of the original equations: $y = x^2 + 1$ or $y = x + 3$.

To find the corresponding values of y it may be easier to substitute into the linear equation.

When $x = -1$ we have:

$y = -1 + 3$

$y = \mathbf{2}$

When $x = 2$ we have:

$y = 2 + 3$

$y = \mathbf{5}$

The solutions for this set of simultaneous equations are:

$x = \mathbf{-1}$, $y = \mathbf{2}$ and $x = \mathbf{2}$, $y = \mathbf{5}$.

We could also have solved this system of equations using the **elimination** method.

Solve:

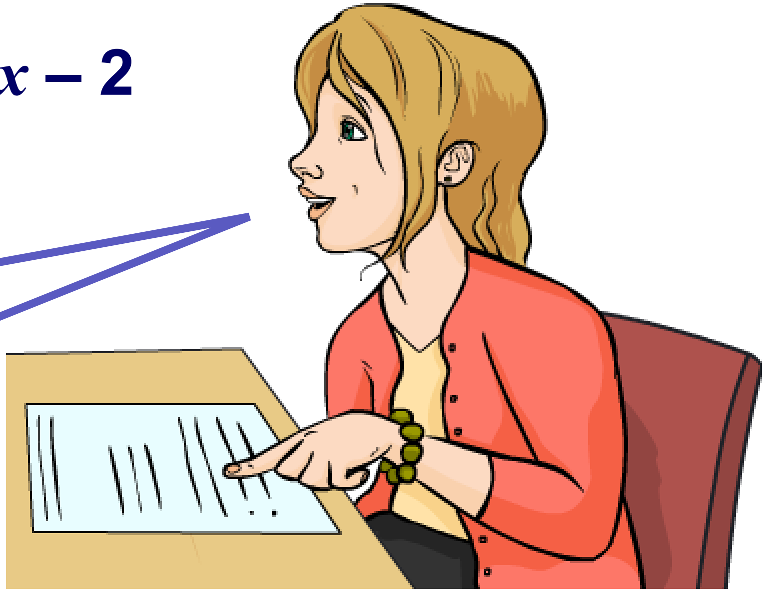
$y = x^2 + 1$ ①

$y = x + 3$ ②

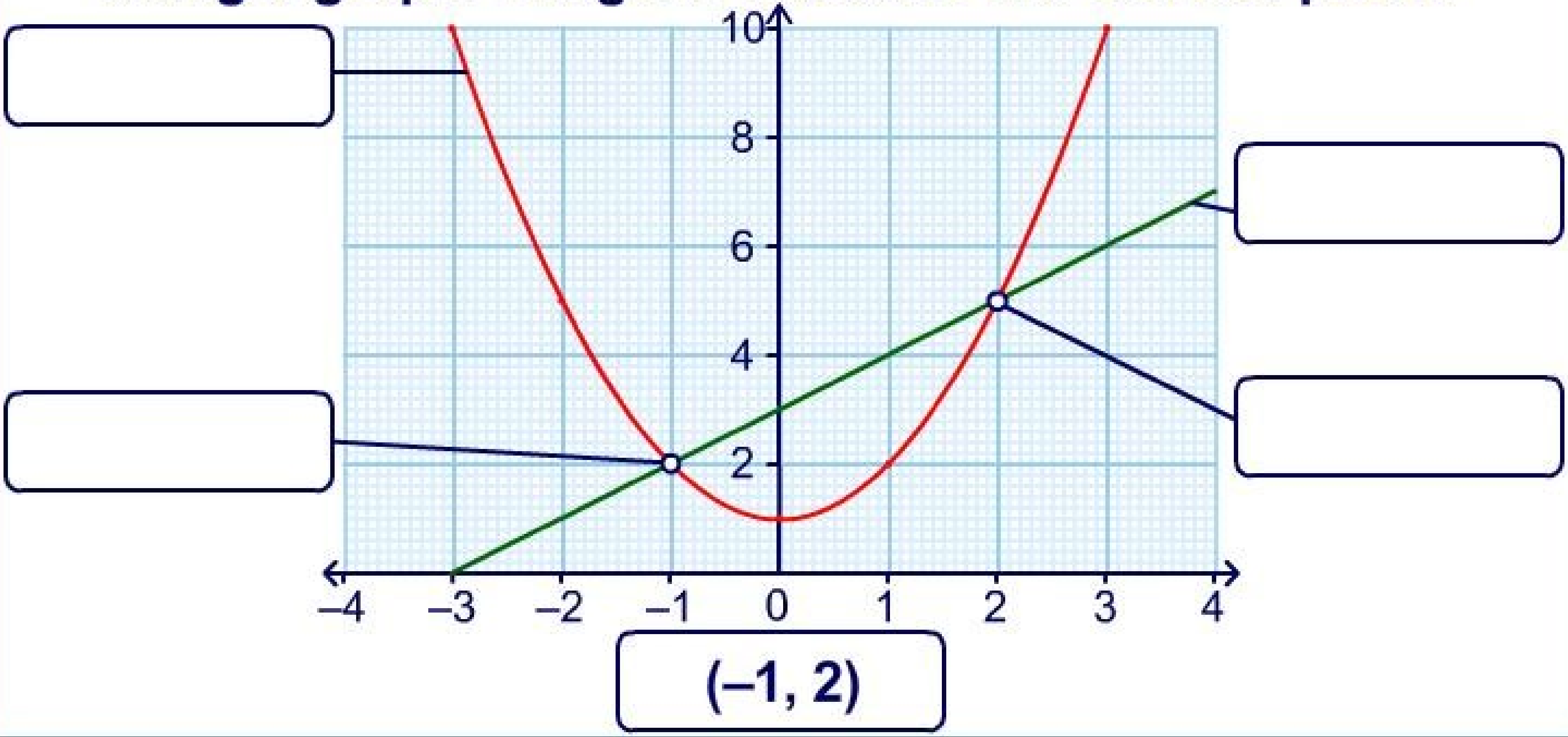
Subtract equation ① from equation ②:

$$0 = x^2 - x - 2$$

This is the same single quadratic equation as the one we found using the substitution method.



We can show the solutions to $y = x^2 + 1$ and $y = x + 3$ using a graph. Drag the labels to the correct place.



Dave hits a ball along a path with height $h = -16t^2 + 15t + 3$ where h is the height in feet and t is the time in seconds since the ball was hit. By chance, the ball hits a balloon released by a child in the crowd at the same time. The balloon's height is given by $h = 3t + 5$.

What height is the balloon when the ball hits it?

by elimination:

$$h = 3t + 5$$
$$-\quad h = -16t^2 + 15t + 3$$
$$0 = 16t^2 - 12t + 2$$

factor and solve for the time:

$$0 = (4t - 1)(4t - 2)$$

$t = ¼$ or $t = ½$

What height is the balloon when the ball hits it?

collision time is given by:

$$0 = (4t - 1)(4t - 2)$$

$$t = \tfrac{1}{4} \quad \text{or} \quad t = \tfrac{1}{2}$$

substitute these values of t into either of the original equations to find h:

$$h = 3t + 5$$

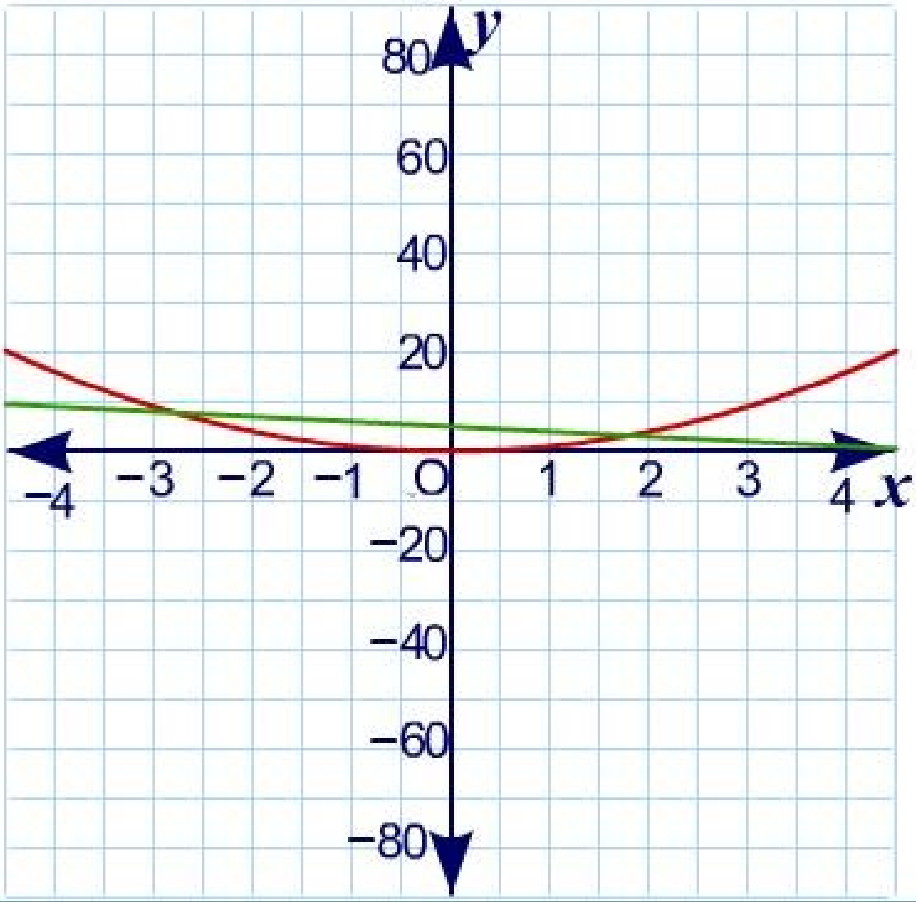
$$h = 3 \times \tfrac{1}{4} + 5 \quad \text{or} \quad h = 3 \times \tfrac{1}{2} + 5$$

= 5.75 feet **= 6.25 feet**

The balloon was at a height of either 5.75 or 6.25 feet when the ball hit it.

Linear and quadratic graphs

Alter the equations to see the intersections change



$1x + 1y = 5$

$y = 1x^2 + 0x + 0$

Intersections:

Using the discriminant

Once we have written two equations as a single quadratic equation, $ax^2 + bx + c = 0$, we can find the **discriminant**, $b^2 - 4ac$, to find how many times the line and the curve intersect and how many solutions the system has.

- When $b2 - 4ac > 0$, there are two distinct points of intersection.
- When $b^2 - 4ac = 0$, there is one point of intersection (or two coincident points). The line is a **tangent** to the curve.
- When $b^2 - 4ac < 0$, there are no points of intersection.

Using the discriminant

Show that the line $y - 4x + 7 = 0$ ①
is a tangent to the curve $y = x^2 - 2x + 2$. ②

Call these equations ① and ②.

rearrange ① to isolate y: $y = 4x - 7$

substitute this expression into ②: $4x - 7 = x^2 - 2x + 2$

rearrange into the usual form: $x^2 - 6x + 9 = 0$

find the discriminant:

$$b^2 - 4ac = (-6)^2 - 4(9)$$
$$= 36 - 36$$
$$= 0$$

$b^2 - 4ac = 0$ and so the line is a tangent to the curve.

A different type of quadratic

Samira finds a pair of simultaneous equations that have a different form: $y = x + 1$ and $x^2 + y^2 = 13$.

What shape is the graph given by $x^2 + y^2 = 13$?

The graph of $x^2 + y^2 = 13$ is a circle with its center at the origin and a radius of $\sqrt{13}$.

We can solve this system of equations algebraically using substitution.

We can also plot the graphs of the equations and observe where they intersect.

A line and a circle

Solve: $y - x = 1$ ①

$x^2 + y^2 = 13$ ②

rearrange ①: $y = x + 1$

substitute into ②: $x^2 + (x + 1)^2 = 13$

expand the parentheses: $x^2 + x^2 + 2x + 1 = 13$

subtract 13 from both sides: $2x^2 + 2x - 12 = 0$

divide all parts by 2: $x^2 + x - 6 = 0$

factor: $(x + 3)(x - 2) = 0$

$x = -3$ or $x = 2$

One linear and one quadratic equation

We can substitute these values of x into one of the equations

$$y = x + 1 \quad (1)$$

$$x^2 + y^2 = 13 \quad (2)$$

to find the corresponding values of y.

It is easiest to substitute into equation (1) because it is linear.

When $x = -3$:

$y = -3 + 1$

$\mathbf{y = -2}$

When $x = 2$:

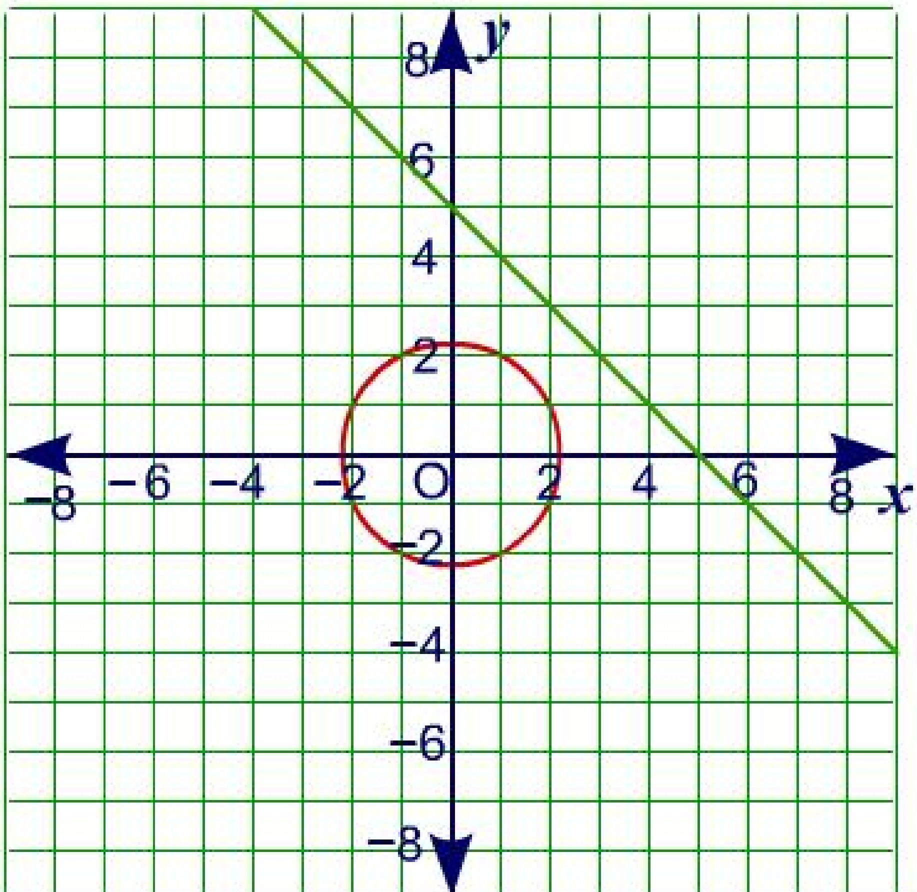
$y = 2 + 1$

$\mathbf{y = 3}$

The solutions are $\mathbf{x = -3, y = -2}$ and $\mathbf{x = 2, y = 3}$.

Linear and circular graphs

Alter the equations to see the intersections change



$1x + 1y = 5$

$x^2 + y^2 = 5$

Intersections: